SMOOTHING EFFECT FOR BOLTZMANN EQUATION WITH FULL-RANGE INTERACTIONS

YEMIN CHEN AND LINGBING HE

ABSTRACT. In this work, we are concerned with the regularities of the solutions to Boltzmann equation with the physical collision kernels for the full range of intermolecular repulsive potentials, $r^{-(p-1)}$ with $p > 2$. We give the new and constructive upper and lower bounds for the collision operator in terms of standard fractional Sobolev norm. As an application, we prove that the strong solutions obtained by Desvillettes & Mouhot [30] to homogeneous Boltzmann equation and classical solutions obtained by Gressman-Strain [36, 37] or Alexandre-Morimoto-Ukai-Xu-Yang [9, 11] for the inhomogeneous Boltzmann equation become immediately smooth with respect to all variables. And as another application, we obtain the global entropy dissipation estimate which is a little stronger than the one of Alexandre-Desvillettes-Villani-Wennberg [5].

CONTENTS

1. INTRODUCTION

In the present work, we continue the study on the smoothness of the solutions to the Boltzmann equation with the collision kernels for the inverse intermolecular potentials $r^{-(p-1)}$ with $p > 2$. It is well known that the Boltzmann equation is a fundamental equation in statistical physics. The readers can refer to [21, 22, 42, 51] and the references therein for the physical background of the equation and also for the mathematical theories for this equation. Mathematically, the Boltmzann equation reads:

$$(1.1) \qquad \partial_t f + v \cdot \nabla_x f = Q(f, f),$$

where $f(t, x, v) \geq 0$ is the (spatially periodic) distribution function in the phase space of collision particles which at time $t \geq 0$ and point $x \in \mathbb{T}^3 = [-\pi, \pi]^3$ move with velocity

1

$v \in \mathbb{R}^3$. The Boltzmann collision operator Q is a bilinear operator which acts only on the velocity variables v, that is,

$$Q(g, f)(v) \overset{\text{def}}{=} \int_{\mathbb{R}^3} \int_{SS^2} B(v - v_*, \sigma)(g'_* f' - g_* f) d\sigma dv_*.$$

Here we use the standard shorthand $f = f(v)$, $g_* = g(v_*)$, $f' = f(v')$, $g'_* = g(v'_*)$ where v', v'_* are given by

$$(1.2) \qquad v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma \ , \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma \ , \qquad \sigma \in SS^{n-1}.$$

We stress that the representation follows the parametrization of the set of solutions of the physical law of elastic collision:

$$\begin{aligned} v + v_* &= v' + v'_*, \\ |v|^2 + |v_*|^2 &= |v'|^2 + |v'_*|^2. \end{aligned}$$

The nonnegative function $B(v - v_*, \sigma)$ in the collision operator is called the Boltzmann collision kernel. It is always assumed to depend only on $|v - v_*|$ and $\langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle$. Usually, we introduce the angle variable θ through $\cos\theta = \langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle$. Without loss of generality, we may assume that $B(v - v_*, \sigma)$ is supported in the set $0 \leq \theta \leq \frac{\pi}{2}$, i.e, $\langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle \geq 0$, for otherwise B can be replaced by its symmetrized form:

$$\bar{B}(v - v_*, \sigma) = [B(v - v_*, \sigma) + B(v - v_*, -\sigma)]\mathbf{1}_{\langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle \geq 0}.$$

Above, $\mathbf{1}_A$ is the characteristic function of the set A. The typical example we have in mind is the case that the interaction potential obeys the inverse repulsive potential which takes the form of

$$\phi(r) = r^{-(p-1)}.$$

According to these potentials, in this paper, we consider the collision kernel satisfying the following assumptions:

Assumption A

- The cross-section $B(v - v_*, \sigma)$ takes a product form as

$$(1.3) \qquad\qquad B(v - v_*, \sigma) = \Phi(|v - v_*|)b(\cos\theta),$$

  where both Φ and b are nonnegative functions.
- The angular function $b(t)$ is not locally integrable and it satisfies

$$(1.4) \qquad K\theta^{-1-2s} \leq \sin\theta b(\cos\theta) \leq K^{-1}\theta^{-1-2s}, \quad \text{with } 0 < s < 1, \ K > 0.$$

- The kinetic factor Φ takes the form

$$(1.5) \qquad\qquad \Phi(|v - v_*|) = |v - v_*|^\gamma,$$

  where the parameter γ verifies that $\gamma + 2s > -1$.

We remark that for inverse repulsive potential, there holds that $\gamma = \frac{p-5}{p-1}$ and $s = \frac{1}{p-1}$. It is easy to check that $\gamma + 4s = 1$ which gives the sense of the assumption $\gamma + 2s > -1$. Generally, the case $\gamma > 0$, $\gamma = 0$, and $\gamma < 0$ correspond to so-called hard, maxwellian, and soft potentials.

There are lots of literatures on the well-posedness problem of the Boltzmann equation, and we will start off by mentioning a brief few. As for the case of Grad's angular cut-off, in 1989, DiPerna and Lions [31] proved the celebrated result: the global existence of renormalized solution to the inhomogeneous Boltzmann equation with arbitrary initial data. Thanks to this breakthrough, based on the new definition of weak solution, the hydrodynamic limit from Boltzmann equation to the equations of fluid mechanics can be considered afterwards, see [34, 44, 45]. Another direction to obtain the global solution is due to the work by Guo [39, 40] and Liu-Yang-Yu [46] who introduce the nonlinear energy method to construct the

classical solutions near the equilibrium. We point out that their approach relies heavily on the analysis of the linearized Boltzmann operator.

As for the non cut-off theory which means physically relevant effects of the angular singularities are considered, it has been made big progress in these years, see [2]. In 1995, Desvillettes in [26] first showed that the solution of the spatially homogeneous non cut-off Kac equation becomes very regular with respect to the velocity variable as soon as the time is strictly positive. This testified the conjecture that when the cross section is concentrating on the grazing collisions, the nonlinear collision operator should behave like a fractional Laplacian in the velocity variable v, see [25, 35, 49]. Later on, Alexandre in [3] formally showed that the smoothness estimates could indeed be deduced from the entropy dissipation $D(g, f)$ defined as

$$D(g, f) = - \int_{\mathbb{R}^6} \int_{SS^2} B(|v - v_*|, \sigma)(g'_* f' - g_* f) \log f \, d\sigma dv dv_*.$$

Lions in [43] proved a functional inequality of the form

$$\|\sqrt{f}\|_{\dot{H}^\alpha(|v| \leq R)} \leq C_R \|f\|_{L_v^1}^\theta (\|f\|_{L_v^1} + D(f, f))^{1-\theta}$$

for $\alpha < s$. Shortly after, the optimal Sobolev exponent s was achieved by Villani [50] and Alexandre [4] but at the price that solution is required to be locally bounded below. In 2000, the work of Alexandre-Desvillettes-Villani-Wennberg [5] showed that usual estimate on the entropy dissipation automatically entails regularization effects:

$$(1.6) \qquad \|\sqrt{f}\|_{H^s(|v| \leq R)}^2 \leq C_{g,R}[D(g, f) + \|g\|_{L_2^1(\mathbb{R}_v^3)}\|f\|_{L_2^1(\mathbb{R}_v^3)}].$$

which indicates that for a given $g \in L_v^1$, the Boltzmann operator behaves as:

$$(1.7) \qquad Q(g, f) = -C_g(-\triangle)^s f + \text{lower order terms}.$$

Moreover, two basic formula such as the cancellation lemma and sub-elliptic coercivity estimates are also given there. Thanks to this breakthrough, Alexandre-Villani in [13, 14] first generalized the renormalized solution with defect measure for Boltzmann equation with long-range interaction and then gave the rigorously justification to the Landau approximation. Another application of the basic tools is to demonstrate the smoothing effect of the classical solutions to the spatially homogeneous Boltzmann equation with regularized potentials [28, 48, 12, 41]. We mention that smoothing behavior is radically different from that of the Boltzmann equation with angular cutoff (see for example [27, 32] and the references therein for precise statements). In this last case, propagation of regularity as well as singularities (in the variable v) occurs, thanks to the properties of the positive part of Boltzmann operator (Cf. [52], [17] and [47]). In 2009, Desvillettes and Mouhot in [30] proved some a priori estimates for the stability and uniqueness for spatial homogeneous Boltzmann equation with long-range interaction, and they also showed the existence for moderate angular singularities.

Recently, Alexandre-Morimoto-Ukai-Xu-Yang [7] introduced the pseudo-differential operator and harmonic analysis to build so-called uncertainty principle to study the hypoellipticity of the kinetic equation. And as the application, they showed the regularizing effects for the linearized Boltzmann equation with non cut-off and linearized Landau equation. Later on, in [8], for the modified kinetic factor, that is,

$$\Phi(|v|) = (1 + |v|^2)^{\frac{1}{2}},$$

based on the pseudo-differential calculus and generalized Bobylev formula(see [15]), they developed the methods to sharpen the upper bound estimate for the Boltzmann collision operator (see also [1, 6]) which helped them not only to establish the local existence theory for the non cut-off inhomogeneous Boltzmann equation with arbitrary initial data and but also to prove the instantaneous smoothness of the solutions. More recently, Gressman-Strain [36, 37, 38] and Alexandre-Morimoto-Ukai-Xu-Yang [9, 10, 11] independently established the global existence of the classical solutions to the Boltzmann equation with long-range of

inverse power intermolecular potentials, r^{-p+1} with $p > 2$ when the initial data are close to the equilibrium. Both of the methods rely on the estimate for the linearized collision operator. Let us give some comments on the work of the coercive estimate. In [36], the authors showed that at the linearized level, the collision operator can be regarded as a fractional Laplacian on a manifold and this manifold depends in an essential way on the collision geometry. More recently, in [38], they provide sharp constructive upper and lower bound estimates for the Boltzmann collision operator. It is shown that under the assumption of high regularity and sufficiently rapid growth of the weight at infinity on the function g, there holds that

$$(1.8) \qquad C_g^1 \|f\|_{\dot{N}^{s,\gamma}}^2 \lesssim -\langle Q(g,f), f\rangle_v + \|f\|_{L_\gamma^2}^2 \lesssim C_g^2 \|f\|_{N^{s,\gamma}}^2,$$

with

$$\|f\|_{N^{s,\gamma}}^2 \overset{\text{def}}{=} \|f\|_{L_{\gamma+2s}^2}^2 + \|f\|_{\dot{N}^{s,\gamma}}^2$$

and

$$\|f\|_{\dot{N}^{s,\gamma}}^2 \overset{\text{def}}{=} \int_{\mathbb{R}^6} (\langle v\rangle\langle v'\rangle)^{\frac{\gamma+2s+1}{2}} \frac{(f'-f)^2}{d(v',v)^{3+2s}} \mathbf{1}_{d(v',v)\leq 1} dv dv'.$$

Here the non-isotropic metric $d(v,v')$ is defined on the "lifted" paraboloid:

$$d(v,v') \overset{\text{def}}{=} \sqrt{|v-v'|^2 + \frac{1}{4}(|v|^2 - |v'|^2)^2}.$$

Moreover, under the same assumption, they prove the global entropy production estimates which is

$$(1.9) \qquad D(g,f) \geq \|\sqrt{f}\|_{\dot{N}^{s,\gamma}}^2 - C_g^3 \|f\|_{L_\gamma^1}.$$

We remark that the norm of $\dot{N}^{s,\gamma}$ is a semi-norm. While in the work of Alexandre-Morimoto-Ukai-Xu-Yang [9, 10], they gave another way to understand the coercivity of the collision operator. Precisely, in contrast to [5], they regarded the quantity

$$\int_{\mathbb{R}^6} \int_{SS^2} |v-v_*|^\gamma b(\cos\theta) e^{-\frac{|v_*|^2}{2}} (f'-f)^2 d\sigma dv dv_*$$

as the new norm instead of standard fractional Sobolev norm to bound the linearized Boltzmann operator. This is key point to construct the global classical solutions of the Boltzmann equation when the initial data are near equilibrium.

In the present work, we are going to investigate the regularities of the solutions to both homogeneous and inhomogeneous Boltzmann equation with the physical collision kernels for the full range of intermolecular repulsive potentials. It can be viewed as a continuation of the recent work [24] where they demonstrated the C^∞ regularizing effect for the full Landau equation. As we known, the main difficulty to prove the smoothing effect for the nonlinear Boltzmann equation comes from the upper and lower bound for the collision operator. The main reason lies in the fact that the Boltzmann operator only involves singular integral behaving like a fractional differential operator but no explicit derivative or pseudo-differential operator occurs.

To overcome the difficulty, motivated by the collision geometry and the standard Littlewood-Paley decomposition, we carry out the new strategy to bound the dual form $\langle Q(g,h), f\rangle_v$. Roughly speaking, in contrast to the previous work [8], by denoting $\mathcal{G} = \langle v\rangle^{N_1} g$, $\mathcal{H} = \langle v\rangle^{N_2} h$ and $\mathcal{F} = \langle v\rangle^{N_3} f$, we first transform $\langle Q(g,h), f\rangle_v$ to the new functional $\langle \mathcal{Q}(\mathcal{G}, \mathcal{H}), \mathcal{F}\rangle_v$. The most convenience of the transformation is that the new factors $\langle v_*\rangle^{-N_1}$, $\langle v\rangle^{-N_2}$ and $\langle v'\rangle^{-N_3}$ which are inside the new functional will absorb the weight coming from the cross-section. Thanks to this design, now we can apply the Littlewood-Paley decomposition to the functions $\mathcal{H}$ and $\mathcal{F}$ to make full use of the cancellation between the different frequency part of them. Combined with the Bernstein's inequality and the proper cut off for the angular, the

upper bound estimate in terms of standard fractional Sobolev norm for the functional is finally obtained which also implies the upper bound for the collision operator by duality. One may check the details in section 2.

Another contribution of the paper lies in the new estimation for the coercivity of the Boltzmann collision operator. We show that for the non Maxwellian potentials, the global sub-elliptic estimate with some weight can be obtained. Roughly, if $\gamma + 2s > 0$ and $\gamma \leq 2$, the estimate for regularizing effect (1.6) can be improved as:

$$\tag{1.10} \|\sqrt{f}\langle v\rangle^{\frac{\gamma}{2}}\|^2_{H^s(\mathbb{R}^3)} \leq C_g[D(g,f) + (\|g\|_{L^1_2(\mathbb{R}^3_v)} + 1)^{C_s}\|f\|_{L^1_2(\mathbb{R}^3_v)}].$$

While for the case of $\gamma + 2s \leq 0$, the similar estimate as (1.10) still can be obtained but at the cost that we have to impose the condition of high integrability (for instance, $L^{\frac{3}{2}+\delta}$ with $\delta > 0$) on the function g which is also observed for the estimate to collision operator (one may check the corresponding theorems for details). We remark that the critical value $\gamma = -2s$ corresponds to the threshold below which there is no spectral gap for the linearized Boltzmann operator. We also point out that comparing to the estimate (1.9), we only use the conserved quantities of Boltzmann equation to capture the smoothing effect in (1.10) in the case of $\gamma + 2s > 0$. One may check the corresponding section for details.

With in hand the upper and lower bound for the collision operator, now we are in a position to state our main results. The first one is concerned with the spatial homogeneous Boltzmann equation which means the distribution function does not depend on the spatial variables, i.e,

$$\partial_t f = Q(f, f).$$

Theorem 1.1. *Let the collision kernel $B(|v - v_*|, \sigma)$ verify the assumption A, and f be the unique solution of the homogeneous Boltzmann equation satisfying the infinite moment estimates, that is, for any $l \in \mathbb{R}^+$,*

$$\tag{1.11} \|f\langle v\rangle^l\|_{L^\infty([0,\infty);L^1(\mathbb{R}^3_v))} < \infty, \quad if \quad \gamma + 2s > 0;$$

or

$$\tag{1.12} \|f\langle v\rangle^l\|_{L^\infty([0,\infty);L^2(\mathbb{R}^3_v))} < \infty, \quad if \quad \gamma + 2s \leq 0.$$

Then for all $t_0 > 0$, the solution f lies in $L^\infty([t_0, \infty); \mathcal{S})$.

Remark 1.1. *Noting the global existence result (for the case of $\gamma+2s > 0$) and local existence result (for the case of $\gamma + 2s \leq 0$) for moderate angular singularity (which means $s < \frac{1}{2}$) by Desvillettes-Mouhot [30], it shows that the result of the Theorem 1.1 is not empty. Actually, following the proof of the Theorem 1.1, we can show that the regularity of the strong solution constructed in Theorem 1.3 by Desvillettes-Mouhot [30] can be propagated which implies that the strong solution is exactly the classical solution when we impose the regularity on the initial datum.*

Remark 1.2. *To our knowledge, it is the first time to prove the smoothing effect of the homogeneous Boltzmann equation for the "true" hard potentials and "true" moderately soft potentials. We also mention that the assumption (1.12) for the case of $\gamma+2s$ can be weakened by*

$$\tag{1.13} \|f\langle v\rangle^l\|_{L^\infty([0,\infty);L^{\frac{3}{2}}(\mathbb{R}^3_v))} < \infty, \quad if \quad \gamma + 2s \leq 0.$$

The main reason lies in the upper and lower bounds for the Boltzmann collision operator. We omit the details here and one may check the corresponding parts in Section 4.

Let us give some comments on the difference between our result with the previous work [28, 48, 12, 41]. In their work, they actually deal with the case of modified kinetic factor Φ which usually takes the form of $(1 + |v - v_*|^2)^{\frac{\gamma}{2}}$. We stress out that this mollification plays the key role in the proof to the smoothing effect of the homogeneous Boltzmann equation. In fact, it will bring them both upper and lower bounds for the collision operator. For the

upper bound, the mollification makes it possible to use integration by parts with respect to v_*. Roughly speaking, by Bobylev's formula, one has

$$-\widehat{\triangle Q(g,h)}(\xi) = \int_{\mathbb{R}^3} \int_{SS^2} \left[\hat{g}(\xi^- + \xi_*)\hat{f}(\xi^+ - \xi_*) - \hat{g}(\xi_*)\hat{f}(\xi - \xi_*) \right] b(\frac{\xi}{|\xi|} \cdot \sigma)\hat{\Phi}(\xi_*)|\xi|^2 d\sigma d\xi,$$

where $\xi^+ = \frac{\xi + |\xi|\sigma}{2}, \xi^- = \frac{\xi - |\xi|\sigma}{2}$. Observing the fact

$$|\xi|^2 = |\xi - \xi_*|^2 + |\xi_*|^2 + 2(\xi - \xi_*) \cdot \xi_*$$

and

$$|\xi|^2 = |\xi^+ - \xi_*|^2 + |\xi_*|^2 + 2(\xi^+ - \xi_*) \cdot \xi_* + |\xi^-|^2,$$

one may expect that the derivative required for g can be transferred to the kinetic factor Φ which leads to the optimal upper bound for the collision operator, that is,

$$\|Q(f,g)\|_{H_N^m} \lesssim \|f\|_{L^1_{N+ + (\gamma+2s)+}} \|g\|_{H^{m+2s}_{(N+\gamma+2s)+}}.$$

While for the lower bound, thanks to the inequality

$$\langle v \rangle^\gamma \leq \langle v - v_* \rangle^\gamma \langle v_* \rangle^{|\gamma|},$$

the coercivity estimate of the collision operator for the case of hard potentials and soft potentials can be concluded to the case of Maxwellian potential. Thus the lower bound for the Boltzmann operator can be easily obtained due to the work by Alexandre-Desvillettes-Villani-Wennberg [5]. Since now the collision kernel only verifies the assumption A, one has to find another approach to give the estimates to the upper and lower bound for the collision operator. And these are exactly what we do in this paper.

For the inhomogeneous Boltzmann equation, to achieve our goal, we still have to bypass the problem how to get the regularity with respect to x, v. Thanks to the upper bound estimate for the collision operator, we show that $Q(g, f)$ belongs to the space $L^2_{t,x}(H_v^{-s})$. This means the hypo-elliptic estimate in [16] for the kinetic equation can be applied. One may treat the Boltzmann equation as Alexandre-Morimoto-Ukai-Xu-Yang done in [7] by employing the generalized uncertainty principle. Here we opt for another approach which mainly comes from the work [24]: once the fractional derivatives(with respect to x) are gained, to avoid estimating the commutator, one may continue to perform the energy estimates (and the estimates based on the averaging lemmas) for weighted finite differences of derivatives of f. By iteration, we finally can obtain the full one derivative with respect to x and v. One has

Theorem 1.2. *Let the collision kernel $B(|v - v_*|, \sigma)$ verifies the assumption A, and f be the unique classical solution of the inhomogeneous Boltzmann equation satisfying that for any $l \in \mathbb{R}^+$,*

(1.14)
$$\|f\langle v \rangle^l\|_{L^\infty([0,\infty);H^5_{x,v})} < \infty,$$

and there exists a universal constant C_l and C_u such that for any $x \in \mathbb{T}^3$,

(1.15)
$$0 < C_l < \|f\|_{L^1(\mathbb{R}^3_v)} < C_u < \infty.$$

Then the solution f lies in $W^{\infty,\infty}([t_0,\infty);H^{\infty,l}_{x,v})$ for all $t_0 > 0$.

Let $\mu = \frac{1}{(2\pi)^{\frac{3}{2}}}e^{-|v|^2}$ be the normalized Maxwillian and $F = F(t, x, v)$ be the standard perturbation with respect to μ as

$$f = \mu + \sqrt{\mu}F.$$

Then by Theorem 1 of [37], we get for any $l \geq 0$ and any integer $N \geq 5$,

$$\|f(t)\langle v \rangle^l\|_{H^N_{x,v}} \leq C_1 + C_2\|F_0\langle v \rangle^l\|_{H^N_{x,v}},$$

where C_1, C_2 depends on l and the constants appeared in Theorem 1 of [37]. Moreover, simple calculation gives that for any $x \in \mathbb{T}^3$,

$$\|\mu\|_{L^1(\mathbb{R}^3)} - \|F\|_{L^\infty}\|\sqrt{\mu}\|_{L^1(\mathbb{R}^3)} \le \|f\|_{L^1(\mathbb{R}^3)} \le \|\mu\|_{L^1(\mathbb{R}^3)} + \|F\|_{L^\infty}\|\sqrt{\mu}\|_{L^1(\mathbb{R}^3)}.$$

Choose $\|F\|_{L^\infty}$ small enough and then we can obtain the estimate (1.15) which implies that the Theorem 1.2 can be applied to the solutions constructed by Gressman-Strain [36, 37] or Alexandre-Morimoto-Ukai-Xu-Yang [9, 11] for the inhomogeneous Boltzmann equation.

The rest of the paper will be organized as follows. First of all, in section 2, we will use Littlewood-Paley analysis to study the upper bound estimate for the collision operator. Moreover, the estimate for the commutator between weight and Boltzmann operator is also given there. In section 3, we will give the proof to the improved coercivity estimate of the collision operator. Then in next two sections, the regularizing effect for homogeneous and inhomogeneous Boltzmann equation will be proven under the initial regularity assumption on the solution. In the appendix, we shall give the proof to some useful interpolation inequality.

Let us complete this section by the function spaces and notations, which we shall use throughout the paper. For notational simplicity, we omit the integrating domains $\mathbb{T}^3$ and $\mathbb{R}^3$, which correspond to variables x and variable v respectively. For example, we write $L^2_{x,v}$ instead of $L^2_x(\mathbb{T}^3; L^2_v(\mathbb{R}^3))$. For integer $N \ge 0$, we define the Sobolev space

$$H^N_{x,v} = \left\{ f(x,v) : \sum_{|\alpha|+|\beta| \le N} \|\partial^\alpha_x \partial^\beta_v f\|_{L^2_{x,v}} < +\infty \right\},$$

and for integer $N \ge 0$ and real number $l \ge 0$, we define the weighted Sobolev space

$$H^{N,l}_{x,v} = \left\{ f(x,v) : \sum_{|\alpha|+|\beta| \le N} \|(\partial^\alpha_x \partial^\beta_v f)\langle v \rangle^l\|_{L^2_{x,v}} < +\infty \right\},$$

where the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and $\partial^\alpha_x = \partial^{\alpha_1}_{x_1} \partial^{\alpha_2}_{x_2} \partial^{\alpha_3}_{x_3}$ with $x = (x_1, x_2, x_3)$, and $\langle v \rangle = (1 + |v|^2)^{\frac{1}{2}}$. The notations for β are the same. It is obvious that $H^{N,0}_{x,v} = H^N_{x,v}$. We also define $H^\infty_{x,v}$ and $H^{\infty,l}_{x,v}$ by

$$H^\infty_{x,v} = \bigcap_{N \ge 0} H^N_{x,v}, \qquad H^{\infty,l}_{x,v} = \bigcap_{N \ge 0} H^{N,l}_{x,v}.$$

We also introduce the standard notations

$$\|f\|_{L^p_\alpha} = \left(\int_{\mathbb{R}^3} |f(v)|^p \langle v \rangle^{\alpha p} dv \right)^{\frac{1}{p}}, \quad \|f\|_{L \log L} = \int_{\mathbb{R}^3} f \log(1 + f) dv,$$

and

$$\|f\|_{H^s_l} = \|f \langle v \rangle^l\|_{H^s}.$$

$a \lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a \le Cb$. $a \sim b$ if both $a \lesssim b$ and $b \lesssim a$.

2. Upper bound on the collision operator

In this section, we shall give the upper bound estimate for the collision operator. Our main motivation comes from the singularity of the cross-section and collision geometry which allow us to apply Littlewood-Paley analysis to the boundedness of the collision operator in terms of weighted fractional Sobolev spaces. It is one of the key steps to prove the smoothing effect of the non cut-off Boltzmann equation. We remark that the variables (t, x) are considered as parameter for the all the estimates in this section.

Theorem 2.1. *Let $0 < s < 1$ and $N_2, N_3 \in \mathbb{R}$. Suppose $N_1 = |N_2| + |N_3|$ and $\tilde{N}_1 = N_2 + N_3$ with $\tilde{N}_1 \ge \gamma + 2s$. Then for nonnegative and smooth functions g, h and f, there hold*

(1) *if $\gamma + 2s > 0$,*

(2.16) $$|\langle Q(g,h), f\rangle_v| \lesssim \|g\|_{L^1_{N_1}(\mathbb{R}^3_v)} \|h\|_{H^s_{N_2}(\mathbb{R}^3_v)} \|f\|_{H^s_{N_3}(\mathbb{R}^3_v)};$$

(2) *if $\gamma + 2s \le 0$,*

(2.17) $$|\langle Q(g,h), f\rangle_v| \lesssim (\|g\|_{L^1_{N_1}(\mathbb{R}^3_v)} + \|g\|_{L^{\frac{3}{2}}_{N_1}(\mathbb{R}^3_v)}) \|h\|_{H^s_{N_2}(\mathbb{R}^3_v)} \|f\|_{H^s_{N_3}(\mathbb{R}^3_v)}.$$

Let us give some comments on the main result of the theorem. First of all, (2.16) and (2.17) can be regarded as another proof to the fact that the Boltzmann operator takes the form of (1.7). Secondly, we stress that the weight in v comes not only from the kinetic factor Φ but also from the integration with respect to the angular. Thirdly, in the case of $\gamma + 2s < 0$, the additional $L^{\frac{3}{2}}$ bound for the function g results from the strong singularity caused by the kinetic factor Φ. Fourthly, by duality, one may take $N_3 = 0$ to obtain the upper bound for the collision operator which will be very useful in the next section. Last we would point out that our proof relies only on the trick of change of variables and cancellation lemma.

Proof of the Theorem 2.1: By change of variables, one may obtain

$$\langle Q(g,h), f\rangle_v = \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_* \int_{SS^2} B(v - v_*, \sigma) g_* h(f' - f) d\sigma$$

$$= \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_* \int_{SS^2} B(v - v_*, \sigma) \langle v_* \rangle^{-N_1} \langle v \rangle^{-N_2} (\langle v_* \rangle^{N_1} g_*)(\langle v \rangle^{N_2} h)$$
$$\times (\langle v' \rangle^{-N_3} (\langle v' \rangle^{N_3} f') - \langle v \rangle^{-N_3} (\langle v \rangle^{N_3} f)) d\sigma.$$

Set $\mathcal{G} = \langle v \rangle^{N_1} g, \mathcal{H} = \langle v \rangle^{N_2} h$ and $\mathcal{F} = \langle v \rangle^{N_3} f$. Then we can rewrite the above equality as

$$\langle Q(g,h), f\rangle_v$$
$$= \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_* \int_{SS^2} B(v - v_*, \sigma) \langle v_* \rangle^{-N_1} \langle v \rangle^{-N_2} \mathcal{G}_* \mathcal{H}(\langle v' \rangle^{-N_3} \mathcal{F}' - \langle v \rangle^{-N_3} \mathcal{F}) d\sigma$$
$$\stackrel{\text{def}}{=} \langle \mathcal{Q}(\mathcal{G}, \mathcal{H}), \mathcal{F}\rangle_v.$$

In the following analysis, we will turn our attention to the new defined functional involving the Boltzmann collision operator. Let us give some comments on the new defined functional. It will bring us two convenience: the first one is that the weight inside the integration will absorb the polynomial of $|v - v_*|$ which probably comes from the cross-section; the second one is that we can use the Littlewood-Paley decomposition for the functions $\mathcal{H}$ and $\mathcal{F}$ which is key for the upper bound estimates.

Set $B \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^3, |\xi| \le \frac{4}{3}\}$ and $\mathcal{C} \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^3, \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$. In view of Littlewood-Paley decomposition, one may introduce two cut off functions $\phi \in C_c^\infty(B)$ and $\varphi \in C_c^\infty(\mathcal{C})$ which satisfy

$$\phi(\xi) + \sum_{j \ge 0} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^3.$$

We denote $h \stackrel{\text{def}}{=} \mathcal{F}^{-1}\varphi$ and $\tilde{h} \stackrel{\text{def}}{=} \mathcal{F}^{-1}\phi$, then the dyadic operators $\triangle_j$ can be defined as follows

$$\triangle_j f = 2^{3j} \int_{\mathbb{R}^3} h(2^j y) f(x - y) dy, \quad \text{for} \quad j \ge 0.$$

$$\triangle_{-1} f = \int_{\mathbb{R}^3} \tilde{h}(y) f(x - y) dy.$$

Then the new defined functional can be presented as

(2.18) $$\langle \mathcal{Q}(\mathcal{G}, \mathcal{H}), \mathcal{F}\rangle_v = \sum_{j < k} \langle \mathcal{Q}(\mathcal{G}, \mathcal{H}_k), \mathcal{F}_j\rangle_v + \sum_{j \le k} \langle \mathcal{Q}(\mathcal{G}, \mathcal{H}_j), \mathcal{F}_k\rangle_v,$$

where $\mathcal{H}_k = \triangle_k \mathcal{H}$ and $\mathcal{F}_j = \triangle_j \mathcal{F}$. Now we will perform the estimate for the Boltzmann collision operator. Since the proof is a little bit longer, we shall divide it into two steps.

Step 1: Frequency dominated by the function $\mathcal{H}$. We first treat with the case that the frequency of function $\mathcal{H}_k$ prevails over the one of the function $\mathcal{F}_j$ which means $j < k$. Introduce the smooth function ϕ defined as before, set $\phi_j(w) = \phi(2^j w)$ and one has that

$$\langle \mathcal{Q}(\mathcal{G}, \mathcal{H}_k), \mathcal{F}_j \rangle_v = \sum_{i=1}^{3} \Gamma_i,$$

where

$$\Gamma_1 \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_* \int_{SS^2} B(v - v_*, \sigma) \langle v_* \rangle^{-N_1} \langle v \rangle^{-N_2} \mathcal{G}_* \mathcal{H}_k \phi_j(|v - v'|)$$
$$\mathbf{1}_{2^{-j}|v-v_*|^{-1} \leq c\frac{\pi}{4}} (\langle v' \rangle^{-N_3} \mathcal{F}_j' - \langle v \rangle^{-N_3} \mathcal{F}_j) d\sigma,$$

$$\Gamma_2 \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_* \int_{SS^2} B(v - v_*, \sigma) \langle v_* \rangle^{-N_1} \langle v \rangle^{-N_2} \mathcal{G}_* \mathcal{H}_k [1 - \phi_j(|v - v'|)]$$
$$\mathbf{1}_{2^{-j}|v-v_*|^{-1} \leq c\frac{\pi}{4}} (\langle v' \rangle^{-N_3} \mathcal{F}_j' - \langle v \rangle^{-N_3} \mathcal{F}_j) d\sigma,$$

and

$$\Gamma_3 \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_* \int_{SS^2} B(v - v_*, \sigma) \langle v_* \rangle^{-N_1} \langle v \rangle^{-N_2} \mathcal{G}_* \mathcal{H}_k$$
$$\mathbf{1}_{2^{-j}|v-v_*|^{-1} \geq c\frac{\pi}{4}} (\langle v' \rangle^{-N_3} \mathcal{F}_j' - \langle v \rangle^{-N_3} \mathcal{F}_j) d\sigma.$$

We remark that the above decomposition comes from the collision geometry and the singularity caused by the angular.

To overcome the strong singularity caused by the collision kernel, motivated by the cancelation lemma, we shall use standard Taylor expansion. Precisely, let

(2.19) $$\mathbb{F}_j = \langle v \rangle^{-N_3} \mathcal{F}_j,$$

then one has

(2.20) $$\mathbb{F}_j(v') - \mathbb{F}_j(v) = (v' - v) \cdot \nabla_v \mathbb{F}_j(v) + \int_0^1 (v' - v) \otimes (v' - v) : \nabla_v^2 \mathbb{F}(\gamma(\kappa)) d\kappa,$$

where $\gamma(\kappa) = \kappa v' + (1 - \kappa) v$.

We stress that we only give the proof to the estimate in the case of $s \geq \frac{1}{2}$ and one may follow the same procedure to prove the case of $s < \frac{1}{2}$.

Lemma 2.1. *If $\gamma + 2s > 0$, there holds*

(2.21) $$|\Gamma_1| \lesssim 2^{2js} \|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} \|\mathcal{H}_k\|_{L^2(\mathbb{R}_v^3)} \|\mathcal{F}_j\|_{L^2(\mathbb{R}_v^3)},$$

and if $\gamma + 2s \leq 0$, there holds

(2.22) $$|\Gamma_1| \lesssim 2^{2js} (\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} + \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)}) \|\mathcal{H}_k\|_{L^2(\mathbb{R}_v^3)} \|\mathcal{F}_j\|_{L^2(\mathbb{R}_v^3)}.$$

Proof: Γ_1 can be split into two parts $\Gamma_{1,1}$ and $\Gamma_{1,2}$ which separately contain the term in the righthand side of (2.20). Notice that

$$\int_{SS^2} b(\langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle)(v - v')\phi_j(|v - v'|)d\sigma$$

$$= \int_{SS^2} b(\langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle)\frac{v - v'}{|v - v'|} \cdot \frac{v - v_*}{|v - v_*|} |v - v'|\phi_j(|v - v'|)\frac{v - v_*}{|v - v_*|} d\sigma$$

$$= \int_{SS^2} b(\langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle)(\frac{1 - \langle \frac{v-v_*}{|v-v_*|}, \sigma \rangle}{2})\phi_j(|v - v'|)d\sigma(v - v_*)$$

Since $\sin\frac{\theta}{2} = \frac{|v-v'|}{|v-v_*|}$, one may obtain that

$$\left| \int_{SS^2} b(\langle \frac{v-v_*}{|v-v_*|}, \sigma \rangle)(v-v')\phi_j(|v-v'|)d\sigma \right|$$

$$\lesssim \int_{\sqrt{\frac{1-\langle\frac{v-v_*}{|v-v_*|},\sigma\rangle}{2}}\lesssim 2^{-j}|v-v_*|^{-1}} b(\langle \frac{v-v_*}{|v-v_*|}, \sigma \rangle) \frac{1-\langle\frac{v-v_*}{|v-v_*|},\sigma\rangle}{2} d\sigma |v-v_*|$$

$$\lesssim \int_0^{2^{-j}|v-v_*|^{-1}} \theta^{1-2s}d\theta |v-v_*|$$

$$\lesssim (2^{-j}|v-v_*|^{-1})^{2-2s}|v-v_*|.$$

Observing that $|v-v_*| \sim |v_*-v'| \sim |v_*-\gamma(\kappa)|$, we may deduce that there holds

$$(2.23) \qquad \langle v_*\rangle^{-N_1}\langle v\rangle^{-N_2}\langle \gamma(\kappa)\rangle^{-N_3} \lesssim \langle v_*-\gamma(\kappa)\rangle^{-\tilde{N}_1}.$$

The reader may check it directly by the definition of N_1 and $\tilde{N}_1$.

Then

$$|\Gamma_{1,1}| = \left| \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_* \int_{SS^2} B(v-v_*,\sigma)\langle v_*\rangle^{-N_1}\langle v\rangle^{-N_2}\mathcal{G}_*\mathcal{H}_k\phi_j(|v-v'|) \right.$$

$$\left. \mathbf{1}_{2^{-j}|v-v_*|^{-1}\leq c\frac{\pi}{4}}(v'-v)\cdot\nabla_v\mathbb{F}_j(v)d\sigma \right|$$

$$\lesssim \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_* (2^{-j}|v-v_*|^{-1})^{1-2s}|v-v_*|^{\gamma+1}\langle v-v_*\rangle^{-\tilde{N}_1}\mathcal{G}_*|\mathcal{H}_k|(|\mathcal{F}_j|+|\nabla_v\mathcal{F}_j|)$$

where we use the fact that $2^{-j}|v-v_*|^{-1} \leq c\frac{\pi}{4}$ and

$$(2.24) \qquad |\nabla_v\mathbb{F}_j(v)| \lesssim \langle v\rangle^{-N_3}(|\mathcal{F}_j|+|\nabla_v\mathcal{F}_j|).$$

For the case of $\gamma+2s > 0$, one may take $\tilde{N}_1 \geq \gamma+2s$ and get

$$\Gamma_{1,1} \lesssim 2^{2js}2^{-j}\|\mathcal{G}\|_{L^1(\mathbb{R}^3_v)}\|\mathcal{H}_k\|_{L^2(\mathbb{R}^3_v)}(\|\mathcal{F}_j\|_{L^2(\mathbb{R}^3_v)}+\|\nabla_v\mathcal{F}_j\|_{L^2(\mathbb{R}^3_v)})$$

$$\lesssim 2^{2js}\|\mathcal{G}\|_{L^1(\mathbb{R}^3_v)}\|\mathcal{H}_k\|_{L^2(\mathbb{R}^3_v)}\|\mathcal{F}_j\|_{L^2(\mathbb{R}^3_v)}.$$

Here we use the Berstein's inequality.

For the case of $-1 < \gamma+2s \leq 0$, one may take $\tilde{N}_1 \geq -|\gamma+2s|$ and obtain the similar estimate

$$\Gamma_{1,1} \lesssim \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_* 2^{-j}2^{2js}(1+|v-v_*|^{\gamma+2s}\mathbf{1}_{|v-v_*|\leq 1})\mathcal{G}_*|\mathcal{H}_k|(|\mathcal{F}_j|+|\nabla_v\mathcal{F}_j|)$$

$$\lesssim 2^{2js}(\|\mathcal{G}\|_{L^1(\mathbb{R}^3_v)}+\|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}^3_v)})\|\mathcal{H}_k\|_{L^2(\mathbb{R}^3_v)}\|\mathcal{F}_j\|_{L^2(\mathbb{R}^3_v)}.$$

Next, we focus on the estimate for $\Gamma_{1,2}$. Notice

$$\Gamma_{1,2} \leq \int_0^1 d\kappa \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_* \int_{SS^2} B(v-v_*,\sigma)\langle v_*\rangle^{-N_1}\langle v\rangle^{-N_2}\mathcal{G}_*|\mathcal{H}_k|\phi_j(|v-v'|)$$

$$\mathbf{1}_{2^{-j}|v-v_*|^{-1}\leq c\frac{\pi}{4}}|v'-v|^2|\nabla_v^2\mathbb{F}_j(\gamma(\kappa))|d\sigma.$$

Note that

$$(2.25) \qquad |\nabla_v^2\mathbb{F}_j(v)| \lesssim \langle v\rangle^{-N_3}(|\mathcal{F}_j|+|\nabla_v\mathcal{F}_j|+|\nabla_v^2\mathcal{F}_j|).$$

Then by Cauchy-Schwartz inequality, it suffices to bound the quantities

$$I \overset{\text{def}}{=} \int_0^1 d\kappa \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_* \int_{SS^2} B(v-v_*,\sigma)\langle v_*\rangle^{-N_1}\langle v\rangle^{-N_2}\mathcal{G}_*\phi_j(|v-v'|)$$

$$\mathbf{1}_{2^{-j}|v-v_*|^{-1}\leq c\frac{\pi}{4}}|v'-v|^2\langle \gamma(\kappa)\rangle^{-N_3}\left(\sum_{i=0}^2 |\nabla_v^i\mathcal{F}_j(\gamma(\kappa))|\right)^2 d\sigma$$

and

$$II \stackrel{\text{def}}{=} \int_0^1 d\kappa \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_* \int_{SS^2} B(v - v_*, \sigma) \langle v_* \rangle^{-N_1} \langle v \rangle^{-N_2} \mathcal{G}_* \phi_j(|v - v'|)$$
$$\mathbf{1}_{2^{-j}|v-v_*|^{-1} \le c\frac{\pi}{4}} |v' - v|^2 \langle \gamma(\kappa) \rangle^{-N_3} |\mathcal{H}_k|^2 d\sigma.$$

Next we follow the well-known change of variables $u = \gamma(\kappa) = \kappa v' + (1 - \kappa)v$, which changes v to u. Thanks to the fact that the Jacobian is

$$(2.26) \qquad \left| \frac{du_m}{dv_n} \right| = (1 - \frac{\kappa}{2})^2 \{(1 - \frac{\kappa}{2}) + \frac{\kappa}{2} \langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle\},$$

one has

$$I \lesssim \int_0^1 d\kappa \int_{\mathbb{R}^3} du \int_{\mathbb{R}^3} dv_* \int_{SS^2} B(v - v_*, \sigma) \langle v_* - u \rangle^{-\tilde{N}_1} \mathcal{G}_* (|\mathcal{F}_j(u)| + |\nabla_v \mathcal{F}_j(u)|$$
$$+ |\nabla_v^2 \mathcal{F}_j(u)|)^2 \mathbf{1}_{2^{-j}|u-v_*|^{-1} \le c\frac{\pi}{4}} \phi_j(|v - v'|)|v' - v|^2 d\sigma,$$

where we use (2.23). Due to the fact that

$$(2.27) \qquad |v' - v|^2 = |v - v_*|^2 \frac{1 - \langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle}{2}$$
$$\lesssim |u - v_*|^2 \frac{1 - \langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle}{2},$$

one may arrive at

$$I \le \int_{\mathbb{R}^3} du \int_{\mathbb{R}^3} dv_* |v_* - u|^{\gamma+2} \langle v_* - u \rangle^{-\tilde{N}_1} \mathcal{G}_* (|\mathcal{F}_j(u)| + |\nabla_v \mathcal{F}_j(u)|$$
$$+ |\nabla_v^2 \mathcal{F}_j(u)|)^2 \mathbf{1}_{2^{-j}|u-v_*|^{-1} \le c\frac{\pi}{4}} \int_{SS^2} b(\langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle) \frac{1 - \langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle}{2} \phi_j(|v - v'|) d\sigma.$$

Noting that

$$\int_{SS^2} b(\langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle) \frac{1 - \langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle}{2} \phi_j(|v - v'|) d\sigma$$
$$\lesssim \int_{\sqrt{\frac{1 - \langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle}{2}} \lesssim 2^{-j}|u-v_*|^{-1}} b(\langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle) \frac{1 - \langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle}{2} d\sigma$$
$$\lesssim \int_0^{2^{-j}|u-v_*|^{-1}} \theta^{1-2s} d\theta \lesssim (2^{-j}|u - v_*|^{-1})^{2-2s},$$

we get

$$I \le 2^{-2j} 2^{2js} \int_{\mathbb{R}^3} du \int_{\mathbb{R}^3} dv_* |v_* - u|^{\gamma+2s} \langle v_* - u \rangle^{-\tilde{N}_1} \mathcal{G}_* (|\mathcal{F}_j(u)| + |\nabla_v \mathcal{F}_j(u)|$$
$$+ |\nabla_v^2 \mathcal{F}_j(u)|)^2,$$

which implies that for the case of $\gamma + 2s > 0$ and $\tilde{N}_1 \ge \gamma + 2s$,

$$I \lesssim 2^{2js} 2^{-2j} \|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} \|\mathcal{F}_j\|_{H^2(\mathbb{R}_v^3)}^2,$$

and for the case of $\gamma + 2s \le 0$ and $\tilde{N}_1 \ge -|\gamma + 2s|$,

$$I \lesssim 2^{2js} 2^{-2j} (\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} + \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)}) \|\mathcal{F}_j\|_{H^2(\mathbb{R}_v^3)}^2.$$

The similar estimate can be applied to II and it gives that for the case of $\gamma + 2s > 0$,

$$II \lesssim 2^{2js} 2^{-2j} \|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} \|\mathcal{H}_k\|_{L^2(\mathbb{R}_v^3)}^2,$$

and for the case of $\gamma + 2s \leq 0$,

$$II \lesssim 2^{2js}2^{-2j}(\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} + \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)})\|\mathcal{H}_k\|_{L^2(\mathbb{R}_v^3)}^2.$$

Then the fact $\Gamma_{1,2} \leq I^{\frac{1}{2}}II^{\frac{1}{2}}$ and the Berstein's inequality lead to the estimate for $\Gamma_{1,2}$ which is exactly as the same as the case of $\Gamma_{1,1}$. We complete the proof to the Lemma. ∎

Lemma 2.2. *If $\gamma + 2s > 0$, there holds*

(2.28) $$|\Gamma_2| \lesssim 2^{2js}\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)}\|\mathcal{H}_k\|_{L^2(\mathbb{R}_v^3)}\|\mathcal{F}_j\|_{L^2(\mathbb{R}_v^3)},$$

and if $\gamma + 2s \leq 0$, there holds

(2.29) $$|\Gamma_2| \lesssim 2^{2js}(\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} + \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)})\|\mathcal{H}_k\|_{L^2(\mathbb{R}_v^3)}\|\mathcal{F}_j\|_{L^2(\mathbb{R}_v^3)}.$$

Proof: Due to the cut-off function ϕ_j, it is easy to check that there is no singularity caused by the collision kernel. Then one may estimate Γ_2 directly and we only present the proof to bound the quantity

$$\Gamma_{2,1} \overset{\text{def}}{=} \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_* \int_{SS^2} B(v - v_*, \sigma)\langle v_*\rangle^{-N_1}\langle v\rangle^{-N_2}\mathcal{G}_*\mathcal{H}_k[1 - \phi_j(|v - v'|)]$$
$$\mathbf{1}_{2^{-j}|v-v_*|^{-1} \leq c\frac{\pi}{4}}\langle v'\rangle^{-N_3}\mathcal{F}_j'd\sigma,$$

By Cauchy-Schwartz inequality, it can be reduced to bound

$$III \overset{\text{def}}{=} \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_* \int_{SS^2} B(v - v_*, \sigma)\langle v_*\rangle^{-N_1}\langle v\rangle^{-N_2}\mathcal{G}_*\mathcal{H}_k^2[1 - \phi_j(|v - v'|)]$$
$$\mathbf{1}_{2^{-j}|v-v_*|^{-1} \leq c\frac{\pi}{4}}\langle v'\rangle^{-N_3}d\sigma,$$

and

$$IV \overset{\text{def}}{=} \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_* \int_{SS^2} B(v - v_*, \sigma)\langle v_*\rangle^{-N_1}\langle v\rangle^{-N_2}\mathcal{G}_*[1 - \phi_j(|v - v'|)]$$
$$\mathbf{1}_{2^{-j}|v-v_*|^{-1} \leq c\frac{\pi}{4}}\langle v'\rangle^{-N_3}|\mathcal{F}_j'|^2d\sigma.$$

We choose to bound the quantity IV since it is a little more complicated than III. Fixed σ and v_*, we perform the change of variables $v \to v'$ and by a direct calculation, its Jacobian determinant is $\frac{1}{8}(1 + \langle\frac{v-v_*}{|v-v_*|}, \sigma\rangle)$ which corresponds to the case $\kappa = 1$ in (2.26). From which together with (2.23), we arrive at

$$IV \lesssim \int_{\mathbb{R}^3} dv' \int_{\mathbb{R}^3} dv_* |v_* - v'|^\gamma \langle v_* - v'\rangle^{-\tilde{N}_1}\mathcal{G}_*\mathbf{1}_{2^{-j}|v'-v_*|^{-1} \leq c\frac{\pi}{4}}$$
$$\times|\mathcal{F}_j'|^2 \int_{SS^2} b(\langle\frac{v - v_*}{|v - v_*|}, \sigma\rangle)[1 - \phi_j(|v - v'|)]d\sigma.$$

Noting that

$$\int_{SS^2} b(\langle\frac{v - v_*}{|v - v_*|}, \sigma\rangle)[1 - \phi_j(|v - v'|)]d\sigma\mathbf{1}_{2^{-j}|v'-v_*|^{-1} \leq c\frac{\pi}{4}}$$
$$\lesssim \int_{\sqrt{\frac{1-\langle\frac{v-v_*}{|v-v_*|},\sigma\rangle}{2}} \gtrsim 2^{-j}|v'-v_*|^{-1}} b(\langle\frac{v - v_*}{|v - v_*|}, \sigma\rangle)d\sigma$$
$$\lesssim \int_{c2^{-j}|v'-v_*|^{-1}}^{\frac{\pi}{2}} \theta^{-1-2s}d\theta \lesssim (2^{-j}|v' - v_*|^{-1})^{-2s},$$

one has

$$IV \lesssim 2^{2js} \int_{\mathbb{R}^3} dv' \int_{\mathbb{R}^3} dv_* |v_* - v'|^{\gamma+2s}\langle v_* - v'\rangle^{-\tilde{N}_1}\mathcal{G}_*|\mathcal{F}_j'|^2.$$

When $\gamma + 2s > 0$, there holds

$$IV \lesssim 2^{2js}\|\mathcal{G}\|_{L^1(\mathbb{R}^3_v)}\|\mathcal{F}_j\|^2_{L^2(\mathbb{R}^3_v)}.$$

While $\gamma + 2s \leq 0$, we obtain that

$$IV \lesssim 2^{2js}(\|\mathcal{G}\|_{L^1(\mathbb{R}^3_v)} + \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}^3_v)})\|\mathcal{F}_j\|^2_{L^2(\mathbb{R}^3_v)}.$$

The fact that $|\Gamma_{2,1}| \lesssim III^{\frac{1}{2}}IV^{\frac{1}{2}}$ implies $\Gamma_{2,1}$ enjoys the same estimate as in the Lemma and it is enough to show that the Lemma also holds true. ∎

Lemma 2.3. *If $\gamma + 2s > 0$, there holds*

(2.30) $$|\Gamma_3| \lesssim 2^{2js}\|\mathcal{G}\|_{L^1(\mathbb{R}^3_v)}\|\mathcal{H}_k\|_{L^2(\mathbb{R}^3_v)}\|\mathcal{F}_j\|_{L^2(\mathbb{R}^3_v)},$$

and if $\gamma + 2s \leq 0$, there holds

(2.31) $$|\Gamma_3| \lesssim 2^{2js}(\|\mathcal{G}\|_{L^1(\mathbb{R}^3_v)} + \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}^3_v)})\|\mathcal{H}_k\|_{L^2(\mathbb{R}^3_v)}\|\mathcal{F}_j\|_{L^2(\mathbb{R}^3_v)}.$$

Proof: Similarly, to overcome the strong singularity caused by collision kernel, we divide Γ_3 into two parts: $\Gamma_{3,1}$ and $\Gamma_{3,2}$ which defined as

$$\Gamma_{3,1} \overset{\text{def}}{=} \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_* \int_{SS^2} B(v - v_*, \sigma)\langle v_*\rangle^{-N_1}\langle v\rangle^{-N_2}\mathcal{G}_*\mathcal{H}_k$$
$$\mathbf{1}_{2^{-j}|v-v_*|^{-1}\geq c\frac{\pi}{4}}(v' - v)\nabla_v\mathbb{F}_jd\sigma,$$

and

$$\Gamma_{3,2} \overset{\text{def}}{=} \int_0^1 d\kappa \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_* \int_{SS^2} B(v - v_*, \sigma)\langle v_*\rangle^{-N_1}\langle v\rangle^{-N_2}\mathcal{G}_*\mathcal{H}_k$$
$$\mathbf{1}_{2^{-j}|v-v_*|^{-1}\geq c\frac{\pi}{4}}(v' - v) \otimes (v' - v) : \nabla_v^2\mathbb{F}_j(\gamma(\kappa))d\sigma,$$

where we use the notation (2.19). Observing that

$$|\int_{SS^2} b(\langle \frac{v - v_*}{|v - v_*|}, \sigma\rangle)(v' - v)d\sigma|\mathbf{1}_{2^{-j}|v-v_*|^{-1}\geq c\frac{\pi}{4}}$$
$$\lesssim |v - v_*|\int_0^{\frac{\pi}{2}} \theta^{1-2s}d\theta\mathbf{1}_{2^{-j}|v-v_*|^{-1}\geq c\frac{\pi}{4}} \lesssim |v - v_*|(2^{-j}|v - v_*|^{-1})^{2-2s}.$$

then for the term $\Gamma_{3,1}$, one has

$$|\Gamma_{3,1}| \lesssim 2^{-2j}2^{2js}\int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_*|v - v_*|^{\gamma+2s-1}\langle v - v_*\rangle^{-\tilde{N}_1}\mathcal{G}_*|\mathcal{H}_k|$$
$$\times\mathbf{1}_{2^{-j}|v-v_*|^{-1}\geq c\frac{\pi}{4}}(|\mathcal{F}_j| + |\nabla_v\mathcal{F}_j|).$$

For the case of $\gamma + 2s > 0$, one may obtain that there exists a constant $\delta \in (0, 1)$ such that

$$|\Gamma_{3,1}| \lesssim 2^{-2j}2^{2js}\int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_*|v - v_*|^{-1+\delta}\mathcal{G}_*|\mathcal{H}_k|\mathbf{1}_{2^{-j}|v-v_*|^{-1}\geq c\frac{\pi}{4}}$$
$$\times 2^{-\frac{j}{2}}|v - v_*|^{-\frac{1}{2}}(|\mathcal{F}_j| + |\nabla_v\mathcal{F}_j|)$$
$$\lesssim 2^{2js}2^{-\frac{5}{2}j}\|\mathcal{G}\|_{L^1(\mathbb{R}^3_v)}\|\mathcal{H}_k\|_{L^2(\mathbb{R}^3_v)}(\|\mathcal{F}_j\|_{L^\infty(\mathbb{R}^3_v)} + \|\nabla_v\mathcal{F}_j\|_{L^\infty(\mathbb{R}^3_v)})$$
$$\lesssim 2^{2js}\|\mathcal{G}\|_{L^1(\mathbb{R}^3_v)}\|\mathcal{H}_k\|_{L^2(\mathbb{R}^3_v)}\|\mathcal{F}_j\|_{L^2(\mathbb{R}^3_v)},$$

where we use the Bernstein's inequality in the last inequality.

While for the case of $\gamma + 2s \leq 0$, we arrives at

$$
\begin{aligned}
|\Gamma_{3,1}| &\lesssim 2^{-2j}2^{2js}\int_{\mathbb{R}^3}dv\int_{\mathbb{R}^3}dv_*|v-v_*|^{\gamma+2s-1}\mathcal{G}_*|\mathcal{H}_k| \\
&\quad \times\mathbf{1}_{2^{-j}|v-v_*|^{-1}\geq c\frac{\pi}{4}}(|\nabla_v\mathcal{F}_j|+|\mathcal{F}_j|) \\
&\lesssim 2^{2js}2^{-2j}\|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)}\|\mathcal{H}_k\|_{L^2(\mathbb{R}_v^3)}(\|\nabla_v\mathcal{F}_j\|_{L^6(\mathbb{R}_v^3)}+\|\mathcal{F}_j\|_{L^6(\mathbb{R}_v^3)}) \\
&\lesssim 2^{2js}\|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)}\|\mathcal{H}_k\|_{L^2(\mathbb{R}_v^3)}\|\mathcal{F}_j\|_{L^2(\mathbb{R}_v^3)}.
\end{aligned}
$$

Next we shall focus on the estimate for $\Gamma_{3,2}$. Thanks to the estimate (2.25), one may obtain

$$
\begin{aligned}
|\Gamma_{3,2}| &\lesssim \int_0^1 d\kappa\int_{\mathbb{R}^3}dv\int_{\mathbb{R}^3}dv_*\int_{SS^2}B(v-v_*,\sigma)\langle v_*\rangle^{-N_1}\langle v\rangle^{-N_2}\mathcal{G}_*|\mathcal{H}_k| \\
&\quad \mathbf{1}_{2^{-j}|v-v_*|^{-1}\geq c\frac{\pi}{4}}|v'-v|^2\langle\gamma(\kappa)\rangle^{-N_3}\sum_{i=0}^2|\nabla_v^i\mathcal{F}_j(\gamma(\kappa))|d\sigma.
\end{aligned}
$$

Due to the Cauchy-Schwartz's inequality, we only have to bound the quantity as

$$
\begin{aligned}
V &\stackrel{\text{def}}{=} \int_0^1 d\kappa\int_{\mathbb{R}^3}dv\int_{\mathbb{R}^3}dv_*\int_{SS^2}B(v-v_*,\sigma)\langle v_*\rangle^{-N_1}\langle v\rangle^{-N_2}\mathcal{G}_* \\
&\quad \mathbf{1}_{2^{-j}|v-v_*|^{-1}\geq c\frac{\pi}{4}}|v'-v|^2\langle\gamma(\kappa)\rangle^{-N_3}(\sum_{i=0}^2|\nabla_v^i\mathcal{F}_j(\gamma(\kappa))|)^2d\sigma,
\end{aligned}
$$

By change of variables from $v \to \gamma(\kappa) = u$ and the fact (2.26) and (2.27), the term V can be bounded as

$$
\begin{aligned}
V &\lesssim \int_{\mathbb{R}^3}du\int_{\mathbb{R}^3}dv_*|v_*-u|^{\gamma+2}\langle u-v_*\rangle^{-\tilde{N}_1}\mathcal{G}_*\mathbf{1}_{2^{-j}|u-v_*|^{-1}\geq c\frac{\pi}{4}} \\
&\quad \times(\sum_{i=0}^2|\nabla_v^i\mathcal{F}_j(u)|)^2\int_{SS^2}b(\langle\frac{v-v_*}{|v-v_*|},\sigma\rangle)\frac{1-\langle\frac{v-v_*}{|v-v_*|},\sigma\rangle}{2}d\sigma \\
&\lesssim \int_{\mathbb{R}^3}du\int_{\mathbb{R}^3}dv_*|v_*-u|^{\gamma+2}\langle u-v_*\rangle^{-\tilde{N}_1}|\mathcal{G}_*|\mathbf{1}_{2^{-j}|u-v_*|^{-1}\geq c\frac{\pi}{4}} \\
&\quad \times(\sum_{i=0}^2|\nabla_v^i\mathcal{F}_j(u)|)^2(2^{-j}|u-v_*|^{-1})^{2-2s}.
\end{aligned}
$$

For the case of $\gamma + 2s > 0$, one may get

$$
V \lesssim 2^{-2j}2^{2js}\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)}\|\mathcal{F}_j\|_{H^2(\mathbb{R}_v^3)}^2.
$$

While for the case of $-1 < \gamma + 2s \leq 0$, we obtain

$$
\begin{aligned}
V &\lesssim 2^{-2j}2^{2js}\int_{\mathbb{R}^3}du\int_{\mathbb{R}^3}dv_*(1+|v_*-u|^{\gamma+2s}\mathbf{1}_{|v_*-u|\leq 1})\mathcal{G}_*|\nabla_v^2\mathcal{F}_j(u)|^2 \\
&\lesssim 2^{-2j}2^{2js}(\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)}+\|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)})\|\mathcal{F}_j\|_{H^2(\mathbb{R}_v^3)}^2.
\end{aligned}
$$

From which, we can deduce that the case of $\gamma + 2s > 0$, there holds

$$
|\Gamma_{3,2}| \lesssim 2^{2js}\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)}\|\mathcal{H}_k\|_{L^2(\mathbb{R}_v^3)}\|\mathcal{F}_j\|_{L^2(\mathbb{R}_v^3)},
$$

and in the case of $\gamma + 2s \leq 0$, there holds

$$
|\Gamma_{3,2}| \lesssim 2^{2js}(\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)}+\|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)})\|\mathcal{H}_k\|_{L^2(\mathbb{R}_v^3)}\|\mathcal{F}_j\|_{L^2(\mathbb{R}_v^3)}.
$$

From which, we complete the proof to the Lemma. ∎

Putting Lemma 2.1, Lemma 2.2 together with Lemma 2.3 , we easily deduce that for the case of $\gamma + 2s > 0$, there holds

$$\sum_{j<k} |\langle \mathcal{Q}(\mathcal{G}, \mathcal{H}_k), \mathcal{F}_j \rangle_v| \lesssim \sum_{j<k} 2^{2js} \|\mathcal{G}\|_{L^1(\mathbb{R}^3_v)} \|\mathcal{H}_k\|_{L^2(\mathbb{R}^3_v)} \|\mathcal{F}_j\|_{L^2(\mathbb{R}^3_v)}$$

$$\lesssim \|\mathcal{G}\|_{L^1(\mathbb{R}^3_v)} \sum_{j<k} 2^{(j-k)s} \|\mathcal{H}_k\|_{H^s(\mathbb{R}^3_v)} \|\mathcal{F}_j\|_{H^s(\mathbb{R}^3_v)}$$

$$(2.32) \qquad \lesssim \|\mathcal{G}\|_{L^1(\mathbb{R}^3_v)} \|\mathcal{H}\|_{H^s(\mathbb{R}^3_v)} \|\mathcal{F}\|_{H^s(\mathbb{R}^3_v)}.$$

Similarly, for the case of $\gamma + 2s \le 0$, there holds there holds

$$(2.33) \qquad \sum_{j<k} |\langle Q(g, h_k), f_j \rangle_v| \lesssim (\|\mathcal{G}\|_{L^1(\mathbb{R}^3_v)} + \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}^3_v)}) \|\mathcal{H}\|_{H^s(\mathbb{R}^3_v)} \|\mathcal{F}\|_{H^s(\mathbb{R}^3_v)}.$$

Step 2: Frequency dominated by the function $\mathcal{F}$. Since the Boltzmann collision operator can be regrades as the bilinear operator, we believe that in some sense there exists symmetric structure inside the operator. We recall that

$$\langle \mathcal{Q}(\mathcal{G}, \mathcal{H}), \mathcal{F} \rangle_v$$
$$= \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_* \int_{SS^2} B(v - v_*, \sigma) \langle v_* \rangle^{-N_1} \langle v \rangle^{-N_2} \mathcal{G}_* \mathcal{H} \left(\langle v' \rangle^{-N_3} \mathcal{F}' - \langle v \rangle^{-N_3} \mathcal{F} \right) d\sigma.$$

It is easy to check that

$$\langle \mathcal{Q}(\mathcal{G}, \mathcal{H}), \mathcal{F} \rangle_v = \langle \mathcal{Q}_1(\mathcal{G}, \mathcal{H}), \mathcal{F} \rangle_v + \langle \mathcal{Q}_2(\mathcal{G}, \mathcal{H}), \mathcal{F} \rangle_v,$$

where

$$\langle \mathcal{Q}_1(\mathcal{G}, \mathcal{H}), \mathcal{F} \rangle_v$$
$$= \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_* \int_{SS^2} B(v - v_*, \sigma) \langle v_* \rangle^{-N_1} \mathcal{G}_* \left(\langle v \rangle^{-N_2} \mathcal{H} - \langle v' \rangle^{-N_2} \mathcal{H}' \right) \langle v' \rangle^{-N_3} \mathcal{F}' d\sigma,$$

and

$$\langle \mathcal{Q}_2(\mathcal{G}, \mathcal{H}), \mathcal{F} \rangle_v$$
$$= \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_* \int_{SS^2} B(v - v_*, \sigma) \langle v_* \rangle^{-N_1} \mathcal{G}_* \left(\langle v' \rangle^{-N_2} \mathcal{H}' \langle v' \rangle^{-N_3} \mathcal{F}' - \langle v \rangle^{-N_2} \mathcal{H} \langle v \rangle^{-N_3} \mathcal{F} \right).$$

We remark that in the case of $j \le k$, $\langle \mathcal{Q}_1(\mathcal{G}, \mathcal{H}_j), \mathcal{F}_k \rangle_v$ enjoys some similar structure as $\langle \mathcal{Q}(\mathcal{G}, \mathcal{H}_k), \mathcal{F}_j \rangle_v$. Precisely, one may follow the same procedure to handle the inner product $\langle \mathcal{Q}_1(\mathcal{G}, \mathcal{H}_j), \mathcal{F}_k \rangle_v$ as we did for $\langle \mathcal{Q}(\mathcal{G}, \mathcal{H}_k), \mathcal{F}_j \rangle_v$. We point out that the main difference lies in the Taylor expansion. If we set $\mathbb{H}_j = \langle v \rangle^{-N_2} \mathcal{H}_j$, then in this case, the Taylor expansion should be taken as

$$(2.34) \quad \mathbb{H}_j(v) - \mathbb{H}_j(v') = (v - v') \cdot \nabla_v \mathbb{H}_j(v') + \int_0^1 (v - v') \otimes (v - v') : \nabla_v^2 \mathbb{H}(\gamma(\kappa)) d\kappa,$$

where $\gamma(\kappa) = \kappa v' + (1 - \kappa)v$.

Another difference comes from the following fact. For each σ and v_*, let $\psi_\sigma(v')$ represents the inverse transform $v' \to \psi_\sigma(v') = v$(see [5]). Then due to (2.26), one has

$$(2.35) \qquad |\frac{dv'}{dv}| = \frac{\langle \frac{v'-v_*}{|v'-v_*|}, \sigma \rangle^2}{4}.$$

Thus for fixed v_* and smooth function ϕ, one has

$$\int_{\mathbb{R}^3} dv \int_{SS^2} B(v - v_*, \sigma) g(v') \phi(|v - v'|)(v - v') d\sigma$$

$$= 4 \int_{\mathbb{R}^3} dv \int_{\langle \frac{v-v_*}{|v-v_*|}, \sigma \rangle \ge \frac{1}{\sqrt{2}}} B(\psi_\sigma(v) - v_*, \sigma) g(v) \phi(|\psi_\sigma(v) - v|) \frac{\psi_\sigma(v) - v}{\langle \frac{v-v_*}{|v-v_*|}, \sigma \rangle^2} d\sigma$$

$$(2.36) \qquad = 0,$$

where we use the symmetric property of $\psi_\sigma(v)$ with respect to σ. The fact will give the reduction for proof to the corresponding terms such as $\Gamma_{1,1}$ and $\Gamma_{3,1}$.

Fortunately, the differences mentioned before do harmless to the proof for $\langle Q_1(\mathcal{G}, \mathcal{H}_j), \mathcal{F}_k\rangle_v$ as we did in the step 1. We omit the details here and obtain that for the case of $\gamma + 2s > 0$, there holds

$$(2.37) \qquad \sum_{j \leq k} |\langle Q_1(\mathcal{G}, \mathcal{H}_j), \mathcal{F}_k\rangle_v| \lesssim \|\mathcal{G}\|_{L^1(\mathbb{R}^3_v)} \|\mathcal{H}\|_{H^s(\mathbb{R}^3_v)} \|\mathcal{F}\|_{H^s(\mathbb{R}^3_v)},$$

and for the case of $\gamma + 2s \leq 0$, there holds

$$(2.38) \qquad \sum_{j \leq k} |\langle Q_1(\mathcal{G}, \mathcal{H}_j), \mathcal{F}_k\rangle_v| \lesssim (\|\mathcal{G}\|_{L^1(\mathbb{R}^3_v)} + \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}^3_v)}) \|\mathcal{H}\|_{H^s(\mathbb{R}^3_v)} \|\mathcal{F}\|_{H^s(\mathbb{R}^3_v)}.$$

Now we turn to estimate $Q_2(\mathcal{G}, \mathcal{H}_j), \mathcal{F}_k\rangle_v$. Actually, by change of variables and cancellation lemma(see [5]), it can be written as

$$\langle Q_2(\mathcal{G}, \mathcal{H}_j), \mathcal{F}_k\rangle_v$$

$$= |SS^1| \int_{\mathbb{R}^6} \int_0^{\frac{\pi}{2}} \sin\theta \left(\frac{1}{\cos^3 \frac{\theta}{2}} B(\frac{|v - v_*|}{\cos \frac{\theta}{2}}, \cos\theta) - B(|v - v_*|, \cos\theta) \right)$$

$$\times \langle v_*\rangle^{-N_1} \mathcal{G}_* \langle v\rangle^{-N_2} \mathcal{H} \langle v\rangle^{-N_3} \mathcal{F} d\theta dv dv_*.$$

It is easy to check that

$$B(\frac{|v - v_*|}{\cos^3 \frac{\theta}{2}}, \cos\theta) - B(|v - v_*|, \cos\theta)$$

$$= b(\cos\theta)|v - v_*|^\gamma \{(\cos^{-1} \frac{\theta}{2})^{\gamma+3} - 1\}.$$

Using the fact that

$$x^m - y^m = m \int_y^x z^{m-1} dz,$$

we obtain that for $\theta \in [0, \frac{\pi}{2}]$,

$$(\cos^{-1} \frac{\theta}{2})^{\gamma+3} - 1 \lesssim \sin^2 \frac{\theta}{2},$$

which immediately implies that

$$\left| \langle Q_2(\mathcal{G}, \mathcal{H}_j), \mathcal{F}_k\rangle_v \right|$$

$$\lesssim \int_{\mathbb{R}^6} \int_0^{\frac{\pi}{2}} \theta^{1-2s} |v - v_*|^\gamma \langle v - v_*\rangle^{-\tilde{N}_1} \mathcal{G}_* |\mathcal{H}_j \mathcal{F}_k| d\theta dv dv_*$$

$$(2.39) \qquad \lesssim \int_{\mathbb{R}^6} |v - v_*|^\gamma \langle v - v_*\rangle^{-\tilde{N}_1} \mathcal{G}_* |\mathcal{H}_j \mathcal{F}_k| dv dv_*$$

$$\overset{\text{def}}{=} \Xi_1 + \Xi_2,$$

where

$$\Xi_1 = \int_{\mathbb{R}^6} |v - v_*|^\gamma \langle v - v_*\rangle^{-\tilde{N}_1} \mathcal{G}_* |\mathcal{H}_j \mathcal{F}_k| \mathbf{1}_{2^{-k}|v-v_*|^{-1} \geq c\frac{\pi}{4}} dv dv_*,$$

and

$$\Xi_2 = \int_{\mathbb{R}^6} |v - v_*|^\gamma \langle v - v_*\rangle^{-\tilde{N}_1} \mathcal{G}_* |\mathcal{H}_j \mathcal{F}_k| \mathbf{1}_{2^{-k}|v-v_*|^{-1} \leq c\frac{\pi}{4}} dv dv_*$$

Here we emphasize that the analogue decomposition for the case $0 < s < \frac{1}{2}$ is radically different from the case of $\frac{1}{2} \leq s < 1$. The quantity $2^{-k}|v - v_*|^{-1}$ inside the decomposition should be replaced by $2^{-j}|v - v_*|^{-1}$. We remark that here j represents the low frequency and k represents the high frequency.

As for the term Ξ_1, one has

$$\Xi_1 \lesssim \int_{\mathbb{R}^6} |v - v_*|^\gamma (2^{-k}|v - v_*|^{-1})^{2-2s} \langle v - v_* \rangle^{-\tilde{N}_1} \mathcal{G}_* |\mathcal{H}_j \mathcal{F}_k| \mathbf{1}_{2^{-k}|v-v_*|^{-1} \geq c\frac{\pi}{4}} dv dv_*$$

$$\lesssim 2^{2ks} 2^{-2k} \int_{\mathbb{R}^6} |v - v_*|^{\gamma+2s-2} \langle v - v_* \rangle^{-\tilde{N}_1} \mathcal{G}_* |\mathcal{H}_j \mathcal{F}_k| \mathbf{1}_{2^{-k}|v-v_*|^{-1} \geq c\frac{\pi}{4}} dv dv_*$$

For the case of $\gamma + 2s > 0$, one has for some small $\delta \in (0, 2)$,

$$|v - v_*|^{\gamma+2s-2} \langle v - v_* \rangle^{-\tilde{N}_1} \leq |v - v_*|^{\delta-2},$$

which leads to

$$\Xi_1 \lesssim 2^{2ks} 2^{-2k} \int_{\mathbb{R}^6} |v - v_*|^{\delta-2} \mathcal{G}_* |\mathcal{H}_j \mathcal{F}_k| \mathbf{1}_{2^{-k}|v-v_*|^{-1} \geq c\frac{\pi}{4}} dv dv_*$$

$$\lesssim 2^{2ks} 2^{-2k} \|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} \|\mathcal{H}_j\|_{L^6(\mathbb{R}_v^3)} \|\mathcal{F}_k\|_{L^6(\mathbb{R}_v^3)}$$

$$\lesssim 2^{-(k-j)(1-s)} \|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} \|\mathcal{H}_j\|_{H^s(\mathbb{R}_v^3)} \|\mathcal{F}_k\|_{H^s(\mathbb{R}_v^3)},$$

where we use the fact $2^{-k}|v - v_*|^{-1} \geq c\frac{\pi}{4}$ and the Bernstein's inequality.

While for the case of $-1 < \gamma + 2s \leq 0$, one has

$$\Xi_1 \lesssim 2^{2ks} 2^{-2k} \int_{\mathbb{R}^6} |v - v_*|^{\gamma+2s-2} \mathcal{G}_* |\mathcal{H}_j \mathcal{F}_k| \mathbf{1}_{|v-v_*| \leq c^{-1}\frac{4}{\pi}} dv dv_*$$

$$\lesssim 2^{2ks} 2^{-2k} \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)} \|\mathcal{H}_j\|_{L^6(\mathbb{R}_v^3)} \|\mathcal{F}_k\|_{L^6(\mathbb{R}_v^3)}$$

$$\lesssim 2^{-(k-j)(1-s)} \|\mathcal{G}\|_{L^2(\mathbb{R}_v^3)} \|\mathcal{H}_j\|_{H^s(\mathbb{R}_v^3)} \|\mathcal{F}_k\|_{H^s(\mathbb{R}_v^3)}.$$

We turn to bound the quantity Ξ_2. In the case of $2^{-k}|v - v_*|^{-1} \leq c\frac{\pi}{4}$, one may deduce that

$$1 \lesssim (2^{-k}|v - v_*|^{-1})^{1-2s}.$$

Then we arrive at

$$\Xi_2 \lesssim 2^{2ks} 2^{-k} \int_{\mathbb{R}^6} |v - v_*|^{\gamma+2s-1} \langle v - v_* \rangle^{-\tilde{N}_1} \mathcal{G}_* |\mathcal{H}_j \mathcal{F}_k| \mathbf{1}_{2^{-j}|v-v_*|^{-1} \geq c\frac{\pi}{4}} dv dv_*$$

For the case of $\gamma + 2s > 0$, one has for some small $\delta \in (0, 1)$,

$$|v - v_*|^{\gamma+2s-1} \langle v - v_* \rangle^{-\tilde{N}_1} \leq |v - v_*|^{\delta-1},$$

which leads to

$$\Xi_2 \lesssim 2^{2ks} 2^{-k} \int_{\mathbb{R}^6} |v - v_*|^{\delta-1} \mathcal{G}_* |\mathcal{H}_j \mathcal{F}_k| dv dv_*$$

$$\lesssim 2^{2ks} 2^{-k} \|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} \|\mathcal{H}_j\|_{L^6(\mathbb{R}_v^3)} \|\mathcal{F}_k\|_{L^2(\mathbb{R}_v^3)}$$

$$\lesssim 2^{-(k-j)(1-s)} \|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} \|\mathcal{H}_j\|_{H^s(\mathbb{R}_v^3)} \|\mathcal{F}_k\|_{H^s(\mathbb{R}_v^3)}.$$

While for the case of $-1 < \gamma + 2s \leq 0$, one has

$$|v - v_*|^{\gamma+2s-1} \langle v - v_* \rangle^{-\tilde{N}_1} \leq 1 + |v - v_*|^{\gamma+2s-1} \mathbf{1}_{|v-v_*| \leq 1}.$$

Then it gives

$$\Xi_2 \lesssim 2^{2ks} 2^{-k} (\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} + \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)})(\|\mathcal{H}_j\|_{L^6(\mathbb{R}_v^3)} + \|\mathcal{H}_j\|_{L^2(\mathbb{R}_v^3)})\|\mathcal{F}_k\|_{L^2(\mathbb{R}_v^3)}$$

$$\lesssim 2^{-(k-j)(1-s)} (\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} + \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)})\|\mathcal{H}_j\|_{H^s(\mathbb{R}_v^3)} \|\mathcal{F}_k\|_{H^s(\mathbb{R}_v^3)}.$$

Patch together the estimates before, we finally obtain that for the case of $\gamma + 2s > 0$, there holds

(2.40)
$$\sum_{j \leq k} |\langle \mathcal{Q}_2(\mathcal{G}, \mathcal{H}_j), \mathcal{F}_k \rangle_v| \lesssim \|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} \|\mathcal{H}\|_{H^s(\mathbb{R}_v^3)} \|\mathcal{F}\|_{H^s(\mathbb{R}_v^3)},$$

and for the case of $\gamma + 2s \leq 0$, there holds

$$(2.41) \qquad \sum_{j \leq k} |\langle \mathcal{Q}_2(\mathcal{G}, \mathcal{H}_j), \mathcal{F}_k \rangle_v| \lesssim (\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} + \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)}) \|\mathcal{H}\|_{H^s(\mathbb{R}_v^3)} \|\mathcal{F}\|_{H^s(\mathbb{R}_v^3)}.$$

Thanks to the (2.32), (2.37) and (2.40), we obtain the first estimate in the Theorem 2.1. Similarly, (2.33), (2.38) and (2.41) imply the second one. ∎

In order to get the regularizing effect of the solutions for the inhomogeneous Boltzmann equation, we still need the estimate for some commutator. Fortunately, we can follow the same idea of the proof to the Theorem and then obtain the corollary:

Corollary 2.1. *Let* $N_1 = |N_2| + |N_3| + \max\{|l-2|, |l-1|\}$ *and* $\tilde{N}_1 = N_2 + N_3$ *with* $N_2, N_3, l \in \mathbb{R}$. *Then if* $\tilde{N}_1 \geq l + \gamma$ *and* $s < \frac{1}{2}$, *one has*

$$(2.42) \qquad |\langle Q(g,h)\langle v \rangle^l, f \rangle_v - \langle Q(g, h\langle v \rangle^l), f \rangle_v|$$
$$\lesssim \|g\|_{L^1_{N_1}(\mathbb{R}_v^3)} \|h\|_{H^\varrho_{N_2}(\mathbb{R}_v^3)} \|f\|_{H^\varrho_{N_3}(\mathbb{R}_v^3)},$$

where $\varrho < s$.
    When $\tilde{N}_1 \geq l - 1 + \gamma + 2s$ *and* $s \geq \frac{1}{2}$, *one has that in the case of* $\gamma + 2s > 0$, *there holds*

$$(2.43) \qquad |\langle Q(g,h)\langle v \rangle^l, f \rangle_v - \langle Q(g, h\langle v \rangle^l), f \rangle_v|$$
$$\lesssim \|g\|_{L^1_{N_1}(\mathbb{R}_v^3)} \|h\|_{H^s_{N_2}(\mathbb{R}_v^3)} \|f\|_{L^2_{N_3}(\mathbb{R}_v^3)}.$$

While in the case of $\gamma + 2s \leq 0$, *there holds*

$$(2.44) \qquad |\langle Q(g,h)\langle v \rangle^l, f \rangle_v - \langle Q(g, h\langle v \rangle^l), f \rangle_v|$$
$$\lesssim (\|g\|_{L^1_{N_1}(\mathbb{R}_v^3)} + \|g\|_{L^{\frac{3}{2}}_{N_1}(\mathbb{R}_v^3)}) \|h\|_{H^s_{N_2}(\mathbb{R}_v^3)} \|f\|_{L^2_{N_3}(\mathbb{R}_v^3)}.$$

Proof: Direct calculation gives that

$$\langle Q(g,h)\langle v \rangle^l, f \rangle_v - \langle Q(g, h\langle v \rangle^l), f \rangle_v$$
$$= \int_{\mathbb{R}^6} dv dv_* \int_{SS^2} B(v - v_*, \sigma) \langle v_* \rangle^{-N_1} \langle v \rangle^{-N_2} \mathcal{G}_* \mathcal{H} \langle v' \rangle^{-N_3} \mathcal{F}'(\langle v' \rangle^l - \langle v \rangle^l) d\sigma.$$

Step 1: Bounds in the case of $s < \frac{1}{2}$. It is easy to check

$$|\langle cQ(g,h)\langle v \rangle^l, f \rangle_v - \langle Q(g, h\langle v \rangle^l), f \rangle_v|$$
$$\lesssim \int_{\mathbb{R}^6} dv dv_* \int_{SS^2} |v - v_*|^{\gamma+1} b(\cos\theta) \sin\theta \langle v - v_* \rangle^{-\tilde{N}_1 + l - 1} \mathcal{G}_* \mathcal{H} \mathcal{F}' d\sigma.$$

By Cauchy-Schwartz's inequality and change of variables, the desired estimate can be reduced to the boundness of the quantity

$$\mathcal{K} = \int_{\mathbb{R}^6} dv dv_* |v - v_*|^{\gamma+1} \langle v - v_* \rangle^{-\tilde{N}_1 + l - 1} \mathcal{G}_* \mathcal{H}^2.$$

Since $\tilde{N}_1 \geq l + \gamma$, we deduce that in the case of $\gamma + 1 \geq 0$, there holds

$$(2.45) \qquad \mathcal{K} \lesssim \|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} \|\mathcal{H}\|^2_{L^2(\mathbb{R}_v^3)}.$$

While in the case of $\gamma + 1 < 0$, one has

$$\mathcal{K} \lesssim \int_{\mathbb{R}^6} dv dv_* (1 + |v - v_*|^{\gamma+1} \mathbf{1}_{|v - v_*| \leq 1}) \mathcal{G}_* \mathcal{H}^2.$$

Thanks to the assumption that $\gamma + 2s + 1 > 0$, there exist a positive δ such that $\gamma + 1 = -(2s - \delta)$ which implies that $\chi(v) = \mathbf{1}_{|v| \leq 1} |v|^{\gamma+1} \in L^{\frac{3-\epsilon}{2s-\delta}}$. Then by Hölder's inequality, one may obtain that

$$\mathcal{K} \lesssim \|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} (\|\mathcal{H}\|^2_{L^2(\mathbb{R}_v^3)} + \|\mathcal{H}^2\|_{L^q(\mathbb{R}_v^3)} \|\chi\|_{L^p(\mathbb{R}_v^3)}),$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $p = \frac{3-\epsilon}{2s-\delta}$. By standard Sobolev's embedding theorem, one has

$$\|\mathcal{H}^2\|_{L^q(\mathbb{R}_v^3)} \leq \|\mathcal{H}\|^2_{H^\varrho(\mathbb{R}_v^3)},$$

where $\varrho = \frac{6s-3\delta}{6-2\epsilon}$. Choose $\epsilon < \frac{3\delta}{2s}$, then we deduce that $\varrho < s$. And we arrive at

$$(2.46) \qquad \mathcal{K} \quad \lesssim \quad \|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} \|\mathcal{H}\|^2_{H^\varrho(\mathbb{R}_v^3)}.$$

Combing (2.45), (2.46) and Cauchy-Schwartz's inequality, we easily obtain the first estimate in the corollary.

Step 2: Bounds in the case of $s \geq \frac{1}{2}$. One has the following decomposition:

$$\langle Q(g,h)\langle v\rangle^l, f\rangle_v - \langle Q(g, h\langle v\rangle^l), f\rangle_v$$
$$\stackrel{\text{def}}{=} \mathfrak{R}_1 + \mathfrak{R}_2,$$

where

$$\mathfrak{R}_1 = \int_{\mathbb{R}^6} dvdv_* \int_{SS^2} B(v-v_*, \sigma)\langle v_*\rangle^{-N_1}\langle v'\rangle^{-\tilde{N}_1}\mathcal{G}_*\mathcal{H}'\mathcal{F}'\left(\langle v'\rangle^l - \langle v\rangle^l\right)d\sigma$$

and

$$\mathfrak{R}_2 = \int_{\mathbb{R}^6} dvdv_* \int_{SS^2} B(v-v_*, \sigma)\langle v_*\rangle^{-N_1}\langle v'\rangle^{-N_3}\mathcal{G}_*\mathcal{F}'\left(\langle v\rangle^{-N_2}\mathcal{H} - \langle v'\rangle^{-N_2}\mathcal{H}'\right)\left(\langle v'\rangle^l - \langle v\rangle^l\right)d\sigma.$$

Firstly, as for the $\mathfrak{R}_1$, we have the following lemma:

Lemma 2.4. *Let $\tilde{N}_1 \geq l - 1 + \gamma + 2s$. In the case of $\gamma + 2s > 0$, there holds*

$$\mathfrak{R}_1 \lesssim \|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} \|\mathcal{H}\|_{H^s(\mathbb{R}_v^3)} \|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}.$$

While in the case of $\gamma + 2s \leq 0$, there holds

$$\mathfrak{R}_1 \lesssim (\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} + \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)})\|\mathcal{H}\|_{H^s(\mathbb{R}_v^3)} \|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}.$$

Proof: To overcome the singularity caused by the collision kernel, we use the Taylor expansion formula up to the order 2:

$$\langle v\rangle^l - \langle v'\rangle^l = (v-v') \cdot \nabla_v(\langle v\rangle^l)|_{v=v'} + \int_0^1 (v-v') \otimes (v-v') : \nabla_v^2(\langle v\rangle^l)|_{v=\gamma(\kappa)}d\kappa,$$

where $\gamma(\kappa) = \kappa v' + (1-\kappa)v$. Set

$$\mathfrak{R}_{1,1} = \int_{\mathbb{R}^6} dvdv_* \int_{SS^2} B(v-v_*, \sigma)\langle v_*\rangle^{-N_1}\langle v'\rangle^{-\tilde{N}_1}\mathcal{G}_*\mathcal{H}'\mathcal{F}'(v'-v)\nabla_v(\langle v'\rangle^l)d\sigma.$$

Then we claim that $\mathfrak{R}_{1,1} = 0$. In fact, by change of variables form $v \to v'$ and fact (2.35), one has

$$\mathfrak{R}_{1,1} = \int_{\mathbb{R}^6} dv'dv_*\langle v_*\rangle^{-N_1}\langle v'\rangle^{-\tilde{N}_1}\mathcal{G}_*\mathcal{H}'\mathcal{F}'\nabla_v(\langle v'\rangle^l)\int_{SS^2}\frac{4}{\langle\frac{v'-v_*}{|v'-v_*|}, \sigma\rangle^2}B(v-v_*, \sigma)(v'-v)d\sigma.$$

Thanks to (2.36), we prove the claim that $\mathfrak{R}_{1,1} = 0$. Next we shall focus on the estimate to the following term:

$$\mathfrak{R}_{1,2} = -\int_0^1 d\kappa \int_{\mathbb{R}^6} dvdv_* \int_{SS^2} B(v-v_*, \sigma)\langle v_*\rangle^{-N_1}\langle v'\rangle^{-\tilde{N}_1}$$
$$\times \mathcal{G}_*\mathcal{H}'\mathcal{F}'(v-v') \otimes (v-v') : \nabla_v^2(\langle v\rangle^l)|_{v=\gamma(\kappa)}d\sigma.$$

It is easy to see that

$$\mathfrak{R}_{1,2} \lesssim \int_0^1 d\kappa \int_{\mathbb{R}^6} dvdv_* \int_{SS^2} B(v-v_*, \sigma)\langle v_*\rangle^{-N_1}\langle v'\rangle^{-\tilde{N}_1}\mathcal{G}_*\mathcal{H}'\mathcal{F}'|v-v'|^2\langle\gamma(\kappa)\rangle^{l-2}d\sigma$$
$$\lesssim \int_{\mathbb{R}^6} dv'dv_* \int_0^{\frac{\pi}{2}} \theta^{1-2s}d\theta\langle v_*-v'\rangle^{-\tilde{N}_1+l-2}\mathcal{G}_*\mathcal{H}'\mathcal{F}'|v_*-v'|^{\gamma+2},$$

where we change the variables from v to v' and use the fact

$$(2.47) \qquad |v - v'|^2 \lesssim |v' - v_*|^2 \frac{1 - \langle \frac{v-v_*}{|v-v_*|}, \sigma \rangle}{2}$$

and

$$\langle v_* \rangle^{-N_1} \langle v' \rangle^{-\tilde{N}_1} \langle \gamma(\kappa) \rangle^{l-2} \lesssim \langle v_* - v' \rangle^{-\tilde{N}_1 + l - 2}.$$

Then we deduce that

$$\mathfrak{R}_{1,2} \lesssim \sum_j \mathfrak{R}_{1,2}^j,$$

where

$$\mathfrak{R}_{1,2}^j \stackrel{\text{def}}{=} \int_{\mathbb{R}^6} dv' dv_* \langle v_* - v' \rangle^{-\tilde{N}_1 + l - 2} \mathcal{G}_* |\mathcal{H}_j'| \mathcal{F}' |v_* - v'|^{\gamma+2}.$$

To bound the term $\mathfrak{R}_{1,2}^j$, we first observe that in the region of $2^{-j} |v' - v_*|^{-1} \geq \pi/4$, there holds

$$\mathfrak{R}_{1,2}^j \lesssim \int_{\mathbb{R}^6} dv' dv_* (2^{-j} |v' - v_*|^{-1})^{2-2s} \langle v_* - v' \rangle^{-\tilde{N}_1 + l - 2} \mathcal{G}_* |\mathcal{H}_j'| \mathcal{F}' |v_* - v'|^{\gamma+2}$$

$$\lesssim 2^{-2j(1-s)} \int_{\mathbb{R}^6} dv' dv_* \langle v_* - v' \rangle^{-\tilde{N}_1 + l - 2} \mathcal{G}_* |\mathcal{H}_j'| \mathcal{F}' |v_* - v'|^{\gamma+2s}.$$

Choose $\tilde{N}_1 \geq l - 1 + \gamma + 2s$. Then for the case of $\gamma + 2s > 0$, we obtain that

$$\mathfrak{R}_{1,2}^j \lesssim 2^{-2(1-s)j} \|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} \|\mathcal{H}_j\|_{L^2(\mathbb{R}_v^3)} \|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}.$$

While in the case of $\gamma + 2s \leq 0$, we deduce that

$$\mathfrak{R}_{1,2}^j \lesssim 2^{-2(1-s)j} (\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} + \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)}) \|\mathcal{H}_j\|_{L^2(\mathbb{R}_v^3)} \|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}.$$

Next we treat the case of $2^{-j} |v' - v_*|^{-1} \leq \pi/4$. One has

$$\mathfrak{R}_{1,2}^j \lesssim \int_{\mathbb{R}^6} dv' dv_* (2^{-j} |v' - v_*|^{-1})^{1-2s} \langle v_* - v' \rangle^{-\tilde{N}_1 + l - 2} \mathcal{G}_* |\mathcal{H}_j'| \mathcal{F}' |v_* - v'|^{\gamma+2}$$

$$\lesssim 2^{2js} 2^{-j} \int_{\mathbb{R}^6} dv' dv_* \langle v_* - v' \rangle^{-\tilde{N}_1 + l - 2} \mathcal{G}_* |\mathcal{H}_j'| \mathcal{F}' |v_* - v'|^{\gamma+2s+1},$$

which implies

$$\mathfrak{R}_{1,2}^j \lesssim 2^{-(1-s)j} \|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} \|\mathcal{H}_j\|_{H^s(\mathbb{R}_v^3)} \|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}.$$

Patch together all the estimates for $\mathfrak{R}_{1,2}^j$ and choose $\tilde{N}_1 \geq l - 1 + \gamma + 2s$, we deduce that in the case of $\gamma + 2s > 0$, there holds

$$\sum_j \mathfrak{R}_{1,2}^j \lesssim \|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} \|\mathcal{H}\|_{H^s(\mathbb{R}_v^3)} \|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}.$$

While in the case of $\gamma + 2s \leq 0$, there holds

$$\sum_j \mathfrak{R}_{1,2}^j \lesssim (\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} + \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)}) \|\mathcal{H}\|_{H^s(\mathbb{R}_v^3)} \|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}.$$

$\blacksquare$

Now we turn to focus on the term $\mathfrak{R}_2$. One has

Lemma 2.5. *Let $\tilde{N}_1 \geq l - 1 + \gamma + 2s$. Then in the case of $\gamma + 2s > 0$, there holds*

$$\mathfrak{R}_2 \lesssim \|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} \|\mathcal{H}\|_{H^s(\mathbb{R}_v^3)} \|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}.$$

While in the case of $\gamma + 2s \leq 0$, there holds

$$\mathfrak{R}_2 \lesssim (\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} + \|\mathcal{G}\|_{L^2(\mathbb{R}_v^3)}) \|\mathcal{H}\|_{H^s(\mathbb{R}_v^3)} \|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}.$$

Proof: We introduce the Littlewood-Paley decomposition and set

$$\mathfrak{R}_2^j \;=\; \int_{\mathbb{R}^6} dvdv_* \int_{SS^2} B(v - v_*, \sigma)\langle v_*\rangle^{-N_1}\langle v'\rangle^{-N_3}$$
$$\times \mathcal{G}_*\mathcal{F}'\left(\langle v\rangle^{-N_2}\mathcal{H}_j - \langle v'\rangle^{-N_2}\mathcal{H}_j' \right)\left(\langle v'\rangle^l - \langle v\rangle^l \right)d\sigma.$$

Then $\mathfrak{R}_2 = \sum_j \mathfrak{R}_2^j$. As done before, we also introduce the angular cut-off function ϕ_j and split $\mathfrak{R}_2^j$ into two parts $\mathfrak{R}_{2,1}^j$ and $\mathfrak{R}_{2,2}^j$ which are defined as

$$\mathfrak{R}_{2,1}^j \;=\; \int_{\mathbb{R}^6} dvdv_* \int_{SS^2} B(v - v_*, \sigma)\langle v_*\rangle^{-N_1}\langle v'\rangle^{-N_3}\mathcal{G}_*\mathcal{F}'\left(\langle v\rangle^{-N_2}\mathcal{H}_j - \langle v'\rangle^{-N_2}\mathcal{H}_j' \right)$$
$$\times \phi_j(|v - v'|)\left(\langle v'\rangle^l - \langle v\rangle^l \right)d\sigma$$

and

$$\mathfrak{R}_{2,2}^j \;=\; \int_{\mathbb{R}^6} dvdv_* \int_{SS^2} B(v - v_*, \sigma)\langle v_*\rangle^{-N_1}\langle v'\rangle^{-N_3}\mathcal{G}_*\mathcal{F}'\left(\langle v\rangle^{-N_2}\mathcal{H}_j - \langle v'\rangle^{-N_2}\mathcal{H}_j' \right)$$
$$\times [1 - \phi_j(|v - v'|)]\left(\langle v'\rangle^l - \langle v\rangle^l \right)d\sigma.$$

We first treat with $\mathfrak{R}_{2,1}^j$. One may check that

$$\mathfrak{R}_{2,1}^j \;\lesssim\; \int_{\mathbb{R}^6} dvdv_* \int_{SS^2} B(v - v_*, \sigma)\langle v_*\rangle^{-N_1}\langle v'\rangle^{-N_3}\mathcal{G}_*\mathcal{F}'|v - v'|^2\phi_j(|v - v'|)$$
$$\times \sum_{i=0}^{1} |\nabla_v^i \mathcal{H}_j(\gamma(\kappa_1))|\langle\gamma(\kappa_1)\rangle^{-N_2}\langle\gamma(\kappa_2)\rangle^{l-1}d\sigma.$$

where $\kappa_1, \kappa_2 \in [0, 1]$. Noticing that

$$(2.48) \qquad \langle v_*\rangle^{-N_1}\langle v'\rangle^{-N_3}\langle\gamma(\kappa_1)\rangle^{-N_2}\langle\gamma(\kappa_2)\rangle^{l-1} \;\lesssim\; \langle v_* - \gamma(\kappa_1)\rangle^{-\tilde{N}_1 + l - 1},$$

we deduce that

$$\mathfrak{R}_{2,1}^j \;\lesssim\; \int_{\mathbb{R}^6} dvdv_* \int_{SS^2} B(v - v_*, \sigma)\langle v_* - \gamma(\kappa_1)\rangle^{-\tilde{N}_1 + l - 1}\mathcal{G}_*\mathcal{F}'|v - v'|^2$$
$$\times \phi_j(|v - v'|)\sum_{i=0}^{1} |\nabla_v^i \mathcal{H}_j(\gamma(\kappa_1))|d\sigma.$$

By Cauchy-Schwartz inequality, $\mathfrak{R}_{2,1}^j$ can be controlled by the quantities

$$VI_1 \;\overset{\text{def}}{=}\; \int_{\mathbb{R}^6} dvdv_* \int_{SS^2} B(v - v_*, \sigma)\langle v_* - \gamma(\kappa_1)\rangle^{-\tilde{N}_1 + l - 1}\mathcal{G}_*|v - v'|^2$$
$$\times \phi_j(|v - v'|)\sum_{i=0}^{1} |\nabla_v^i \mathcal{H}_j(\gamma(\kappa_1))|^2 d\sigma$$

and

$$VI_2 \;\overset{\text{def}}{=}\; \int_{\mathbb{R}^6} dvdv_* \int_{SS^2} B(v - v_*, \sigma)\langle v_* - \gamma(\kappa_1)\rangle^{-\tilde{N}_1 + l - 1}\mathcal{G}_*|v - v'|^2$$
$$\times \phi_j(|v - v'|)|\mathcal{F}'|^2 d\sigma.$$

We only need to show the estimate for VI_1. One has

$$VI_1 \lesssim \int_{\mathbb{R}^6} dv dv_* \langle v_* - \gamma(\kappa_1)\rangle^{-\tilde{N}_1+l-1} \mathcal{G}_* |v_* - \gamma(\kappa_1)|^{\gamma+2} \sum_{i=0}^{1} |\nabla_v^i \mathcal{H}_j(\gamma(\kappa_1))|^2$$

$$\times \int_{SS^2} \phi_j(|v-v'|) b(\langle \frac{v-v_*}{|v-v_*|}, \sigma \rangle) \frac{1 - \langle \frac{v-v_*}{|v-v_*|}, \sigma \rangle}{2} d\sigma.$$

Fixed σ and v_* and noting the fact (2.26), we change of variables from $v \to u = \gamma(\kappa_1)$ and then it gives

$$VI_1 \lesssim \int_{\mathbb{R}^6} du dv_* \langle v_* - u\rangle^{-\tilde{N}_1+l-1} \mathcal{G}_* |v_* - u|^{\gamma+2} \sum_{i=0}^{1} |\nabla_v^i \mathcal{H}_j(u)|^2$$

$$\times \int_0^{c2^{-j}|u-v_*|^{-1}} \theta^{1-2s} d\theta.$$

Since $\tilde{N}_1 \geq l - 1 + \gamma + 2s$, then for the case of $\gamma + 2s > 0$, there holds

$$VI_1 \lesssim \|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} \|\mathcal{H}_j\|_{H^s(\mathbb{R}_v^3)}^2.$$

While for the case of $\gamma + 2s \leq 0$, there holds

$$VI_1 \lesssim (\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} + \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)}) \|\mathcal{H}_j\|_{H^s(\mathbb{R}_v^3)}^2.$$

Similar calculation can be applied to VI_2 and it gives that in the case of $\gamma + 2s > 0$, there holds

$$VI_2 \lesssim 2^{-2j(1-s)} \|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} \|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}^2.$$

While for the case of $\gamma + 2s \leq 0$, there holds

$$VI_2 \lesssim 2^{-2j(1-s)} (\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} + \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)}) \|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}^2.$$

From which, we deduce that in the case of $\gamma + 2s > 0$, there holds

$$(2.49) \qquad \sum_j \mathfrak{R}_{2,1}^j \lesssim \|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} \|\mathcal{H}\|_{H^s(\mathbb{R}_v^3)} \|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}.$$

While for the case of $\gamma + 2s \leq 0$, there holds

$$(2.50) \qquad \sum_j \mathfrak{R}_{2,1}^j \lesssim (\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} + \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)}) \|\mathcal{H}\|_{H^s(\mathbb{R}_v^3)} \|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}.$$

Now we treat with the term $\mathfrak{R}_{2,2}^j$. We first observe that the angular function $b(\theta)$ now is locally integrable thanks to the cut-off function $1 - \phi_j(|v-v'|)$. Then the bound for $\mathfrak{R}_{2,2}^j$ can be reduced to the estimation of

$$VII \stackrel{\text{def}}{=} \int_{\mathbb{R}^6} dv dv_* \int_{SS^2} B(v-v_*,\sigma) \langle v_*\rangle^{-N_1} \langle v'\rangle^{-N_3} \mathcal{G}_* \mathcal{F}' \langle v\rangle^{-N_2} |\mathcal{H}_j|$$

$$\times [1 - \phi_j(|v-v'|)] \left(\langle v'\rangle^l - \langle v\rangle^l \right) d\sigma.$$

It is easy to check that

$$VII \lesssim \int_{\mathbb{R}^6} dv dv_* \int_{SS^2} B(v-v_*,\sigma) \langle v_*\rangle^{-N_1} \langle v'\rangle^{-N_3} \mathcal{G}_* \mathcal{F}' \langle v\rangle^{-N_2} |\mathcal{H}_j| |v-v'|$$

$$\times [1 - \phi_j(|v-v'|)] \langle \gamma(\kappa_3)\rangle^{l-1} d\sigma,$$

where $\kappa_3 \in [0,1]$. Thanks to (2.48), by Cauchy-Schwartz inequality, VII can be controlled by VII_1 and VII_2 defined as

$$VII_1 \overset{\text{def}}{=} \int_{\mathbb{R}^6} dvdv_* \int_{SS^2} B(v-v_*, \sigma)\langle v_* - v\rangle^{-\tilde{N}_1 + l - 1}\mathcal{G}_*$$
$$\times |\mathcal{H}_j|^2 |v - v'|[1 - \phi_j(|v - v'|)]d\sigma$$

and

$$VII_2 \overset{\text{def}}{=} \int_{\mathbb{R}^6} dvdv_* \int_{SS^2} B(v-v_*, \sigma)\langle v_* - v'\rangle^{-\tilde{N}_1 + l - 1}\mathcal{G}_*$$
$$\times |\mathcal{F}'|^2 |v - v'|[1 - \phi_j(|v - v'|)]d\sigma.$$

We only need to give the estimate to one of them. As for VII_2, by change of variables from v to v', one may has

$$VII_2 \lesssim \int_{\mathbb{R}^6} dv'dv_* \langle v_* - v'\rangle^{-\tilde{N}_1 + l - 1}|v_* - v'|^{\gamma + 1}\mathcal{G}_*|\mathcal{F}'|^2 \mathbf{1}_{c2^{-j}|v_* - v'|^{-1} < \frac{\pi}{4}}$$
$$\times \int_{\sqrt{1 - \langle\frac{v - v_*}{|v - v_*|}, \sigma\rangle} \geq c2^{-j}|v_* - v'|^{-1}} b(\langle\frac{v - v_*}{|v - v_*|}, \sigma\rangle)\sqrt{1 - \langle\frac{v - v_*}{|v - v_*|}, \sigma\rangle}d\sigma,$$

where we use the fact (2.47). From which, we deduce that

$$VII_2 \lesssim 2^{-j}2^{2js} \int_{\mathbb{R}^6} dv'dv_* \langle v_* - v'\rangle^{-\tilde{N}_1 + l - 1}|v_* - v'|^{\gamma + 2s}\mathcal{G}_*|\mathcal{F}'|^2,$$

which implies that in the case of $\gamma + 2s > 0$, there holds

$$VII_2 \lesssim 2^{-j(1-s)}2^{js}\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)}\|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}^2.$$

While for the case of $\gamma + 2s \leq 0$, there holds

$$VII_2 \lesssim 2^{-j(1-s)}2^{js}(\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} + \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)})\|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}^2.$$

Similarly, one can obtain that in the case of $\gamma + 2s > 0$, there holds

$$VII_1 \lesssim 2^{-j(1-s)}2^{js}\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)}\|\mathcal{H}_j\|_{L^2(\mathbb{R}_v^3)}^2.$$

While for the case of $\gamma + 2s \leq 0$, there holds

$$VII_1 \lesssim 2^{-j(1-s)}2^{js}(\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} + \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)})\|\mathcal{H}_j\|_{L^2(\mathbb{R}_v^3)}^2.$$

Thanks to the Bernstein's inequality, we arrive at in the case of $\gamma + 2s > 0$ and $\tilde{N}_1 \geq l - 1 + \gamma + 2s$, there holds

$$VII \lesssim 2^{-j(1-s)}\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)}\|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}\|\mathcal{H}_j\|_{H^s(\mathbb{R}_v^3)}.$$

While for the case of $\gamma + 2s \leq 0$, there holds

$$VII \lesssim 2^{-j(1-s)}(\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} + \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)})\|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}\|\mathcal{H}_j\|_{H^s(\mathbb{R}_v^3)}.$$

Finally, we obtain that in the case of $\gamma + 2s > 0$, there holds

$$\sum_j \mathfrak{R}_{2,2}^j \lesssim \|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)}\|\mathcal{H}\|_{H^s(\mathbb{R}_v^3)}\|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}.$$

While for the case of $\gamma + 2s \leq 0$, there holds

$$\sum_j \mathfrak{R}_{2,2}^j \lesssim (\|\mathcal{G}\|_{L^1(\mathbb{R}_v^3)} + \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)})\|\mathcal{H}\|_{H^s(\mathbb{R}_v^3)}\|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}.$$

From which together with (2.49) and (2.50), we complete the proof of the lemma. ∎

Thanks to the Lemma 2.4 and Lemma 2.5, we complete the proof to the corollary. ∎

3. Coercivity estimate for the collision operator

In this section, we shall give the coercivity bound for the collision operator. Our main motivation comes from the sub-elliptic estimate (1.6) entailed by entropy dissipation and the upper bound estimate (2.16) and (2.17) for the collision operator. Roughly, our strategy is carried out as follows. We first use the trick to reformulate the functional $\langle -Q(g,f), f\rangle_v$ as $\langle -\mathcal{Q}(\mathcal{G},\mathcal{F}),\mathcal{F}\rangle_v$ by introducing $\mathcal{G} = g\langle v\rangle^N$ and $\mathcal{F} = f\langle v\rangle^N$. As a consequence, the kinetic part $\Phi(|v - v_*|)$ will be cancelled by the additional factor $\langle v_*\rangle^{-N}$ and $\langle v\rangle^{-N}$ which means the cases of the hard potential and soft potential can be reduced to the Maxwellian case but at the price of occurring the lower order terms. We point out that here the lower order term means the lower derivative term or the term with lower weight. Thanks to the estimates (1.6), (2.16) and (2.17), we finally obtain the following theorem:

Theorem 3.1. *Let the collision kernel $B(|v - v_*|, \sigma)$ satisfies the Assumption A. Suppose the function g satisfies*

$$\|g\|_{L^1_2(\mathbb{R}^3_v)} + \|g\|_{L\log L(\mathbb{R}^3_v)} < \infty.$$

Then there exists a constant C_g depending on $\|g\|_{L^1(\mathbb{R}^3_v)}$ and $\|g\|_{L\log L(\mathbb{R}^3_v)}$ such that in the case of $\gamma + 2s > 0$, there holds

$$\langle -Q(g,f), f\rangle_v \;\gtrsim\; C_g\|f\langle v\rangle^{\frac{\gamma}{2}}\|^2_{H^s(\mathbb{R}^3_v)} - (C_g^{1-2s}\|g\langle v\rangle^{\tilde{\gamma}}\|^{2s}_{L^1(\mathbb{R}^3_v)} + \|g\langle v\rangle^{\tilde{\gamma}}\|_{L^1(\mathbb{R}^3_v)})$$

$$(3.51) \qquad\qquad \times\|f\langle v\rangle^{\frac{\gamma}{2}}\|^2_{L^2(\mathbb{R}^3_v)} - \|g\langle v\rangle^{|\gamma|}\|_{L^1(\mathbb{R}^3_v)}\|f\langle v\rangle^{\frac{\gamma}{2}}\|^2_{H^\varrho(\mathbb{R}^3_v)},$$

where $\varrho < s$ and

$$\tilde{\gamma} = |\gamma + 2|\mathbf{1}_{\gamma\le0} + |\gamma - 2|\mathbf{1}_{\gamma>0};$$

and in the case of $\gamma + 2s \le 0$, there holds

$$\langle -Q(g,f), f\rangle_v \;\gtrsim\; C_g\|f\langle v\rangle^{\frac{\gamma}{2}}\|^2_{H^s(\mathbb{R}^3_v)} - (C_g^{1-2s}\|g\langle v\rangle^{|\gamma+2|}\|^{2s}_{L^1(\mathbb{R}^3_v)} + \|g\langle v\rangle^{|\gamma+2|}\|_{L^1(\mathbb{R}^3_v)})$$

$$(3.52) \qquad\qquad \times\|f\langle v\rangle^{\frac{\gamma}{2}}\|^2_{L^2(\mathbb{R}^3_v)} - (\|g\langle v\rangle^{|\gamma|}\|_{L^1(\mathbb{R}^3_v)} + \|g\langle v\rangle^{|\gamma|}\|_{L^{\frac{3}{2}}(\mathbb{R}^3_v)})\|f\langle v\rangle^{\frac{\gamma}{2}}\|^2_{H^\varrho(\mathbb{R}^3_v)},$$

with $\varrho < s$.

Before the proof, let us give some comments on the Theorem 3.1. First of all, the proof of theorem can be applied to obtain the smoothing effect estimate (1.10) which is entailed by entropy dissipation. Secondly, comparing to the upper bound estimates (2.16) and (2.17), we lose $2s$ order of weight in v in the coercivity estimate. The main reason may lie in the fact that it is still not clear to the structure of the collision operator.

Proof of the Theorem: It is easy to see that

$$\langle Q(g,f), f\rangle_v = \int_{\mathbb{R}^6} dv_* dv \int_{SS^2} B(|v - v_*|, \sigma)g_* f(f' - f)d\sigma$$

$$= \frac{1}{2}\int_{\mathbb{R}^6} dv_* dv \int_{SS^2} B(|v - v_*|, \sigma)g_*(f'^2 - f^2)d\sigma$$

$$\qquad -\frac{1}{2}\int_{\mathbb{R}^6} dv_* dv \int_{SS^2} B(|v - v_*|, \sigma)g_*(f' - f)^2 d\sigma$$

$$= \mathfrak{I}_1 - \mathfrak{I}_2.$$

Step 1: Upper bound for $\mathfrak{I}_1$. By change of variables, one has

$$\mathfrak{I}_1 = |SS^1|\int_{\mathbb{R}^6}\int_0^{\frac{\pi}{2}} \sin\theta\left(\frac{1}{\cos^3\frac{\theta}{2}}B(\frac{|v - v_*|}{\cos\frac{\theta}{2}}, \cos\theta) - B(|v - v_*|, \cos\theta)\right)g_* f^2 d\theta dv_* dv$$

$$= |SS^1|\int_{\mathbb{R}^6}\int_0^{\frac{\pi}{2}} \sin\theta b(\cos\theta)|v - v_*|^\gamma\{(\cos^{-1}\frac{\theta}{2})^{\gamma+3} - 1\}g_* f^2 d\theta dv_* dv.$$

Using the fact that

$$x^m - y^m = m \int_y^x z^{m-1} dz,$$

we obtain that for $\theta \in [0, \frac{\pi}{2}]$,

$$(\cos^{-1} \frac{\theta}{2})^{\gamma+3} - 1 \lesssim \sin^2 \frac{\theta}{2},$$

which immediately implies

$$\mathfrak{I}_1 = C \int_{\mathbb{R}^6} |v - v_*|^\gamma g_* f^2 \, dv dv_*.$$

In the forthcoming argument, we shall give the different upper bounds for $\mathfrak{I}_1$ with respect to the value γ.

Case 1: $\gamma > 0$. It is easy to check

$$\mathfrak{I}_1 \lesssim \|g\|_{L^1_\gamma(\mathbb{R}^3_v)} \|f\|^2_{L^2_{\frac{\gamma}{2}}(\mathbb{R}^3_v)}$$

Case 2: $\gamma < 0$. We may rewrite $\mathfrak{I}_1$ as

$$\mathfrak{I}_1 = C \int_{\mathbb{R}^6} |v - v_*|^\gamma \langle v \rangle^{-\gamma} \langle v_* \rangle^\gamma \mathcal{G}_* \mathcal{F}^2 \, dv dv_*,$$

where $\mathcal{G}_* = g_* \langle v \rangle^{-\gamma}, \mathcal{F} = f \langle v \rangle^{\frac{\gamma}{2}}$. It is easy to check that

$$\mathfrak{I}_1 \lesssim \int_{\mathbb{R}^6} (1 + |v - v_*|^\gamma \mathbf{1}_{|v-v_*| \leq 1}) \mathcal{G}_* \mathcal{F}^2 \, dv dv_*.$$

Now we first treat the case $\gamma + 2s > 0$. That is, there exist a positive δ such that $\gamma = -(2s - \delta)$ which implies that $\chi(v) = \mathbf{1}_{|v| \leq 1} |v|^\gamma \in L^{\frac{3-\epsilon}{2s-\delta}}$. Then by Hölder's inequality, one may obtain that

$$\mathfrak{I}_1 \lesssim \|\mathcal{G}\|_{L^1(\mathbb{R}^3_v)} (\|\mathcal{F}\|^2_{L^2(\mathbb{R}^3_v)} + \|\mathcal{F}^2\|_{L^q(\mathbb{R}^3_v)} \|\chi\|_{L^p(\mathbb{R}^3_v)}),$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $p = \frac{3-\epsilon}{2s-\delta}$. By standard Sobolev's embedding theorem, one has

$$\|\mathcal{F}^2\|_{L^q(\mathbb{R}^3_v)} \leq \|\mathcal{F}\|^2_{H^\varrho(\mathbb{R}^3_v)},$$

where $\varrho = \frac{6s - 3\delta}{6 - 2\epsilon}$. Choose $\epsilon < \frac{3\delta}{2s}$, then one may obtain that $\varrho < s$. And we arrive at

$$\mathfrak{I}_1 \lesssim \|g \langle v \rangle^{-\gamma}\|_{L^1(\mathbb{R}^3_v)} \|f \langle v \rangle^{\frac{\gamma}{2}}\|^2_{H^\varrho(\mathbb{R}^3_v)}.$$

Secondly, we handle with the case $-1 < \gamma + 2s < 0$. Let $\delta = 1 + \gamma + 2s$. Then by the Assumption A, it gives that $0 < \delta < 1$ and $\chi(v) \in L^{\frac{3-\epsilon}{1+2s-\delta}}$. Choose $\epsilon < \frac{3\delta}{1+2s}$. Then we can deduce that there exists a constant $\varrho = \frac{6s - 3\delta + \epsilon}{2(3-\epsilon)}$ verifying $\varrho < s$ such that

$$\frac{1 + 2\varrho}{3} = \frac{1 + 2s - \delta}{3 - \epsilon}.$$

Then by Hölder's inequality and Young's inequality, one has that

$$\mathfrak{I}_1 \lesssim \|\mathcal{G}\|_{L^1(\mathbb{R}^3_v)} \|\mathcal{F}\|^2_{L^2(\mathbb{R}^3_v)} + \|\mathcal{F}^2\|_{L^q(\mathbb{R}^3_v)} \|\mathcal{G} * \chi\|_{L^p(\mathbb{R}^3_v)}),$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{2} = \frac{1}{2q} + \frac{\varrho}{3}$. By standard Sobolev's embedding theorem, we obtain that

$$\mathfrak{I}_1 \lesssim (\|\mathcal{G}\|_{L^1(\mathbb{R}^3_v)} + \|\mathcal{G}\|_{L^{\frac{3}{2}}(\mathbb{R}^3_v)}) \|\mathcal{F}\|^2_{H^\varrho(\mathbb{R}^3_v)}.$$

Putting together the estimates for $\mathfrak{I}_1$, we arrive at there exists a constant ϱ verifying $\varrho < s$ such that for the case of $\gamma + 2s > 0$, there holds

(3.53) $$\mathfrak{I}_1 \lesssim \|g \langle v \rangle^{|\gamma|}\|_{L^1(\mathbb{R}^3_v)} \|f \langle v \rangle^{\frac{\gamma}{2}}\|^2_{H^\varrho(\mathbb{R}^3_v)},$$

and for the case of $\gamma + 2s < 0$, there holds

$$(3.54) \qquad \mathfrak{I}_1 \lesssim (\|g\langle v\rangle^{|\gamma|}\|_{L^1(\mathbb{R}_v^3)} + \|g\langle v\rangle^{|\gamma|}\|_{L^{\frac{3}{2}}(\mathbb{R}_v^3)})\|f\langle v\rangle^{\frac{\gamma}{2}}\|_{H^\varrho(\mathbb{R}_v^3)}^2.$$

Step 2: Lower bound for $\mathfrak{I}_2$. Recalling that

$$2\mathfrak{I}_2 = \int_{\mathbb{R}^6}\int_{SS^2} |v - v_*|^\gamma b(\cos\theta)g_*(f' - f)^2 d\sigma dv_* dv,$$

and setting $\mathcal{F} = f\langle v\rangle^{\frac{\gamma}{2}}$, we can rewrite $\mathfrak{I}_2$ as

$$2\mathfrak{I}_2 = \int_{\mathbb{R}^6}\int_{SS^2} |v - v_*|^\gamma b(\cos\theta)g_*(\langle v'\rangle^{-\frac{\gamma}{2}}\mathcal{F}' - \langle v\rangle^{-\frac{\gamma}{2}}\mathcal{F})^2 d\sigma dv_* dv.$$

We shall give the different lower bound for $\mathfrak{I}_2$ with respect to the value γ.

Case 1: $\gamma \le 0$. Thanks to the fact $(A - B)^2 \ge \frac{A^2}{2} - B^2$, one has

$$(\langle v'\rangle^{-\frac{\gamma}{2}}\mathcal{F}' - \langle v\rangle^{-\frac{\gamma}{2}}\mathcal{F})^2$$

$$= \left(\langle v'\rangle^{-\frac{\gamma}{2}}(\mathcal{F}' - \mathcal{F}) + \mathcal{F}(\langle v'\rangle^{-\frac{\gamma}{2}} - \langle v\rangle^{-\frac{\gamma}{2}})\right)^2$$

$$\ge \frac{1}{2}\langle v'\rangle^{-\gamma}(\mathcal{F}' - \mathcal{F})^2 - \mathcal{F}^2(\langle v'\rangle^{-\frac{\gamma}{2}} - \langle v\rangle^{-\frac{\gamma}{2}})^2.$$

Then one may obtain that

$$2\mathfrak{I}_2 \ge \frac{1}{2}\int_{\mathbb{R}^6}\int_{SS^2} |v - v_*|^\gamma b(\cos\theta)g_*\langle v'\rangle^{-\gamma}(\mathcal{F}' - \mathcal{F})^2 d\sigma dv_* dv$$

$$- \int_{\mathbb{R}^6}\int_{SS^2} |v - v_*|^\gamma b(\cos\theta)g_*\mathcal{F}^2(\langle v'\rangle^{-\frac{\gamma}{2}} - \langle v\rangle^{-\frac{\gamma}{2}})^2 d\sigma dv_* dv$$

$$\stackrel{\text{def}}{=} \mathfrak{L}_1 - \mathfrak{L}_2.$$

Noting that

$$|v - v_*|^\gamma\langle v'\rangle^{-\gamma}\langle v_*\rangle^{-\gamma} \gtrsim 1,$$

one has

$$\mathfrak{L}_1 \gtrsim \int_{\mathbb{R}^6} b(\cos\theta)(g_*\langle v_*\rangle^\gamma)(\mathcal{F}' - \mathcal{F})^2 d\sigma dv_* dv.$$

Due to the well-known entropy dissipation inequality, we arrive at

$$\mathfrak{L}_1 \gtrsim C_g\|\mathcal{F}\|_{H^s(\mathbb{R}_v^3)}^2,$$

where C_g depends on $\|g\langle v\rangle^\gamma\|_{L_1^1(\mathbb{R}_v^3)}$, $\|g\langle v\rangle^\gamma\|_{L\log L(\mathbb{R}_v^3)}$ and b. Next, we turn to the estimate of $\mathfrak{L}_2$. One may have

$$\mathfrak{L}_2 \lesssim \int_{\mathbb{R}^6}\int_{SS^2} |v - v_*|^\gamma b(\cos\theta)g_*\mathcal{F}^2|v - v'|^2\langle\gamma(\kappa)\rangle^{-\gamma-2} d\sigma dv_* dv$$

with $\kappa \in [0, 1]$. Noting the fact

$$|v - v'| = |v - v_*|\sin\frac{\theta}{2},$$

we deduce that in the case of $\gamma + 2 \ge 0$, there holds

$$\mathfrak{L}_2 \lesssim \int_{\mathbb{R}^6} g_*\mathcal{F}^2\langle v_*\rangle^{\gamma+2} dv_* dv$$

$$\lesssim \|g\langle v\rangle^{|\gamma+2|}\|_{L^1(\mathbb{R}_v^3)}\|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}^2.$$

From which, we obtain that for $\gamma + 2 \ge 0$, there holds

$$(3.55) \qquad \mathfrak{I}_2 \gtrsim \frac{C_g}{2}\|\mathcal{F}\|_{H^s(\mathbb{R}_v^3)}^2 - \|g\langle v\rangle^{|\gamma+2|}\|_{L^1(\mathbb{R}_v^3)}\|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}^2.$$

While in the case of $\gamma + 2 < 0$, thanks to the Assumption A, one has

$$\gamma + 2 > -1, s > \frac{1}{2}.$$

Then for any ϵ, there holds

$$\mathfrak{L}_2 \lesssim \int_{\mathbb{R}^6}(1 + |v - v_*|^{\gamma+2}\mathbf{1}_{|v-v_*|\leq 1})g_*\mathcal{F}^2\langle v_*\rangle^{|\gamma+2|}dv_*dv$$

$$\lesssim \|g\langle v\rangle^{|\gamma+2|}\|_{L^1(\mathbb{R}_v^3)}(\|\mathcal{F}\|^2_{L^2(\mathbb{R}_v^3)} + \|\mathcal{F}\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}_v^3)})$$

$$\lesssim \|g\langle v\rangle^{|\gamma+2|}\|_{L^1(\mathbb{R}_v^3)}[(1 + \epsilon^{-(2s-1)})\|\mathcal{F}\|^2_{L^2(\mathbb{R}_v^3)} + \epsilon\|\mathcal{F}\|^2_{\dot{H}^s(\mathbb{R}_v^3)}].$$

From which, we obtain that for $\gamma + 2 < 0$, there holds

$$(3.56) \qquad \mathfrak{I}_2 \gtrsim \frac{C_g}{2}\|\mathcal{F}\|^2_{H^s(\mathbb{R}_v^3)} - C_g^{1-2s}(\|g\langle v\rangle^{|\gamma+2|}\|_{L^1(\mathbb{R}_v^3)} + C_g)^{2s}\|\mathcal{F}\|^2_{L^2(\mathbb{R}_v^3)}.$$

Thanks to (3.55) and (3.56), we conclude that for $\gamma < 0$, $\mathfrak{I}_2$ can be estimated as

$$\mathfrak{I}_2 \gtrsim \frac{C_g}{2}\|\mathcal{F}\|^2_{H^s(\mathbb{R}_v^3)} - (C_g^{1-2s}\|g\langle v\rangle^{|\gamma+2|}\|^{2s}_{L^1(\mathbb{R}_v^3)} + \|g\langle v\rangle^{|\gamma+2|}\|_{L^1(\mathbb{R}_v^3)})$$

$$(3.57) \qquad \times\|\mathcal{F}\|^2_{L^2(\mathbb{R}_v^3)}.$$

Case 2: $\gamma > 0$. Observing the fact that

$$\langle v\rangle \sim |v| + \mathbf{1}_{|v|\leq 1},$$

we may obtain that

$$\mathfrak{I}_2 \gtrsim \int_{\mathbb{R}^6}\int_{SS^2}\langle v - v_*\rangle^\gamma b(\cos\theta)g_*(f' - f)^2 d\sigma dv_*dv$$

$$- \int_{\mathbb{R}^6}\int_{SS^2}\mathbf{1}_{|v-v_*|\leq 1}b(\cos\theta)g_*(f' - f)^2 d\sigma dv_*dv.$$

Noting

$$\langle v - v_*\rangle^\gamma \geq \langle v' - v_*\rangle^\gamma \geq \langle v_*\rangle^{-\gamma}\langle v'\rangle^\gamma,$$

and following the similar trick as the one in the case of $\gamma < 0$, we arrive at

$$\mathfrak{I}_2 \gtrsim \frac{1}{2}\int_{\mathbb{R}^6}\int_{SS^2}b(\cos\theta)(g_*\langle v_*\rangle^{-\gamma})(\mathcal{F}' - \mathcal{F})^2 d\sigma dv_*dv$$

$$- \int_{\mathbb{R}^6}\int_{SS^2}b(\cos\theta)g_*\langle v_*\rangle^{-\gamma}f^2(\langle v'\rangle^{\frac{\gamma}{2}} - \langle v\rangle^{\frac{\gamma}{2}})^2 d\sigma dv_*dv$$

$$- \int_{\mathbb{R}^6}\int_{SS^2}\mathbf{1}_{|v-v_*|\leq 1}b(\cos\theta)g_*(f' - f)^2 d\sigma dv_*dv$$

$$\overset{\text{def}}{=} \mathfrak{L}_3 - \mathfrak{L}_4 - \mathfrak{L}_5.$$

Here we also set $\mathcal{F} = f\langle v\rangle^{\frac{\gamma}{2}}$.

It is easy to check that

$$\mathfrak{L}_3 \gtrsim C_g\|\mathcal{F}\|^2_{H^s(\mathbb{R}_v^3)},$$

where C_g depends on $\|g\langle v\rangle^{-\gamma}\|_{L_1^1(\mathbb{R}_v^3)}$, $\|g\langle v\rangle^{-\gamma}\|_{L\log L(\mathbb{R}_v^3)}$ and b. As for the term $\mathfrak{L}_4$, one has

$$\mathfrak{L}_4 \lesssim \int_{\mathbb{R}^6}\int_{SS^2}b(\cos\theta)g_*f^2|v - v'|^2\langle v_*\rangle^{-\gamma}\langle\gamma(\kappa)\rangle^{\gamma-2}d\sigma dv_*dv$$

with $\kappa \in [0, 1]$. From which, we deduce that

$$\mathfrak{L}_4 \lesssim \|g\langle v\rangle^{|\gamma-2|}\|_{L^1(\mathbb{R}_v^3)}\|\mathcal{F}\|^2_{L^2(\mathbb{R}_v^3)}.$$

As for the term $\mathcal{L}_5$, noting that

$$\int_{\mathbb{R}^6} \mathbf{1}_{|v-v_*|\leq 1} b(\cos\theta) g_*(f'-f)^2 d\sigma dv_* dv$$

$$= -2 \int_{\mathbb{R}^6} \mathbf{1}_{|v-v_*|\leq 1} b(\cos\theta) g_* f(f'-f) d\sigma dv_* dv$$

$$+ \int_{\mathbb{R}^6} \mathbf{1}_{|v-v_*|\leq 1} b(\cos\theta) g_*(f'^2-f^2) d\sigma dv_* dv,$$

we conclude that by the proof of the Theorem 2.1 and the result of the step 1, there holds

$$\mathfrak{L}_5 \lesssim \|g\|_{L^1(\mathbb{R}_v^3)}(\|f\|_{L^2(\mathbb{R}_v^3)}^2 + \|f\|_{H^s(\mathbb{R}_v^3)}^2).$$

From which, we deduce that

$$\mathfrak{I}_2 \gtrsim C_g \|\mathcal{F}\|_{H^s(\mathbb{R}_v^3)}^2 - \|g\langle v\rangle^{|\gamma-2|}\|_{L^1(\mathbb{R}_v^3)}\|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}^2 - \|g\|_{L^1(\mathbb{R}_v^3)}\|f\|_{\dot{H}^s(\mathbb{R}_v^3)}^2.$$

Noticing that for any smooth function χ_R defined as $\chi_R = \chi(\frac{\cdot}{R})$ with $0 \leq \chi \leq 1, \chi = 1$ on B_1 and supp$(\chi) \subset B_2$, there holds

$$\|f\|_{\dot{H}^s(\mathbb{R}_v^3)} \leq \|f\chi_R\|_{\dot{H}^s(\mathbb{R}_v^3)} + R^{-\frac{\gamma}{2}}\|\mathcal{F}\|_{\dot{H}^s(\mathbb{R}_v^3)} + \|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}$$

(one may check the proof in the Appendix), which implies that

$$\mathfrak{I}_2 \gtrsim (C_g - R^{-\gamma}\|g\|_{L^1(\mathbb{R}_v^3)})\|\mathcal{F}\|_{H^s(\mathbb{R}_v^3)}^2$$

$$- \|g\langle v\rangle^{|\gamma-2|}\|_{L^1(\mathbb{R}_v^3)}\|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}^2 - \|g\|_{L^1(\mathbb{R}_v^3)}\|f\chi_R\|_{\dot{H}^s(\mathbb{R}_v^3)}^2.$$

Thanks to the entropy dissipation inequality, we can also deduce that for any $R > 0$, there holds

$$\mathfrak{I}_2 + \|g\|_{L^1(\mathbb{R}_v^3)}\|f^2\|_{L^1(\mathbb{R}_v^3)} \gtrsim C_{g,R}\|f\chi_R\|_{\dot{H}^s(\mathbb{R}_v^3)}^2,$$

where $C_{g,R}$ depends on $\|g\|_{L^1(\mathbb{R}_v^3)}$, $\|g\|_{L\log L(\mathbb{R}_v^3)}$, b and R. Choose R_1 such that

$$\|g\|_{L^1(\mathbb{R}_v^3)} \leq \frac{1}{2} C_g R_1^\gamma,$$

then we finally obtain that for $\gamma > 0$, there holds

$$\mathfrak{I}_2 \gtrsim \frac{C_g C_{g,R_1}}{2(\|g\|_{L^1(\mathbb{R}_v^3)} + C_{g,R_1})}\|\mathcal{F}\|_{H^s(\mathbb{R}_v^3)}^2$$

(3.58)
$$- \|g\langle v\rangle^{|\gamma-2|}\|_{L^1(\mathbb{R}_v^3)}\|\mathcal{F}\|_{L^2(\mathbb{R}_v^3)}^2.$$

Now we can conclude that (3.53), (3.57) and (3.58) imply the coercivity estimate for the case of $\gamma + 2s > 0$. And (3.54) and (3.57) imply the coercivity estimate for the case of $\gamma + 2s \leq 0$ which completes the proof to the Theorem 3.1. ∎

As a direct application, we obtain the entropy dissipation estimate:

Theorem 3.2. *Let the collision kernel $B(|v - v_*|, \sigma)$ satisfies the Assumption A. Suppose the function g satisfies*

$$\|g\|_{L_2^1(\mathbb{R}_v^3)} + \|g\|_{L\log L(\mathbb{R}_v^3)} < \infty.$$

Then there exists a constant C_g depending on $\|g\|_{L_1^1(\mathbb{R}_v^3)}$ and $\|g\|_{L\log L(\mathbb{R}_v^3)}$ and a constant $\varrho < s$ such that in the case of $\gamma + 2s > 0$ and $\gamma \leq 2$, there holds

$$\langle D(g,f), f\rangle_v + (C_g^{1-2s}\|g\|_{L_2^1(\mathbb{R}_v^3)}^{2s} + \|g\|_{L_2^1(\mathbb{R}_v^3)} + C_g^{-\frac{\varrho}{s-\varrho}}\|g\|_{L_2^1(\mathbb{R}_v^3)}^{\frac{s}{s-\varrho}})\|f\|_{L_2^1}$$

(3.59)
$$\gtrsim C_g\|\sqrt{f}\langle v\rangle^{\frac{\gamma}{2}}\|_{H^s(\mathbb{R}_v^3)}^2.$$

Proof: Direct calculation gives that

$$D(g,f) \geq -\int_{\mathbb{R}^6} dv_* dv \int_{SS^2} B(|v-v_*|,\sigma)g_*(f'-f)d\sigma$$

$$+ \int_{\mathbb{R}^6} dv_* dv \int_{SS^2} B(|v-v_*|,\sigma)g_*(\sqrt{f'}-\sqrt{f})^2 d\sigma.$$

We stress that the estimates for righthand side of the above inequality are exactly as the same as the ones for $\mathfrak{I}_1$ and $\mathfrak{I}_2$. Then we arrive at

$$D(g,f) \gtrsim C_g\|\sqrt{f}\langle v\rangle^{\frac{\gamma}{2}}\|^2_{H^s(\mathbb{R}_v^3)} - (C_g^{1-2s}\|g\langle v\rangle^{\tilde{\gamma}}\|^{2s}_{L^1(\mathbb{R}_v^3)} + \|g\langle v\rangle^{\tilde{\gamma}}\|_{L^1(\mathbb{R}_v^3)})$$

$$(3.60) \qquad \times\|\sqrt{f}\langle v\rangle^{\frac{\gamma}{2}}\|^2_{L^2(\mathbb{R}_v^3)} - \|g\langle v\rangle^{|\gamma|}\|_{L^1(\mathbb{R}_v^3)}\|\sqrt{f}\langle v\rangle^{\frac{\gamma}{2}}\|^2_{H^\varrho(\mathbb{R}_v^3)},$$

where $\varrho < s$ and

$$\tilde{\gamma} = |\gamma+2|\mathbf{1}_{\gamma\leq 0} + |\gamma-2|\mathbf{1}_{\gamma>0}.$$

Since now $\gamma + 2s > 0$ and $\gamma \leq 2$, we deduce that $\tilde{\gamma} < 2$ and $|\gamma| \leq 2$. Thanks to the Young's inequality

$$\|f\|_{H^\varrho} \lesssim \epsilon\|f\|_{H^s} + \epsilon^{-\frac{\varrho}{s-\varrho}}\|f\|_{L^2},$$

we can rewrite (3.60) as

$$D(g,f) \gtrsim C_g\|\sqrt{f}\langle v\rangle^{\frac{\gamma}{2}}\|^2_{H^s(\mathbb{R}_v^3)} - (C_g^{1-2s}\|g\|^{2s}_{L^1_2(\mathbb{R}_v^3)} + \|g\|_{L^1_2(\mathbb{R}_v^3)})\|f\|_{L^1_2(\mathbb{R}_v^3)}$$

$$-C_g^{-\frac{\varrho}{s-\varrho}}\|g\|^{\frac{s}{s-\varrho}}_{L^1_2(\mathbb{R}_v^3)}\|f\|_{L^1_2(\mathbb{R}_v^3)},$$

which completes the proof to the Theorem. ∎

4. Smoothing effect for the homogeneous Boltzmann equation

In this section, we shall give the proof to the Theorem 1.1.

Proof of the Theorem 1.1: We first assume that infinite L^2 moment estimate (1.12) holds true for all the collision kernel. The inductive argument will be applied to prove the smoothing effect of the homogeneous Boltzmann equation.

Let us assume that for some $m \in \mathbb{N}$ and all $l \in N$, there hold

$$(4.61) \qquad\qquad \sup_{[t_0,\infty)} \|f\|_{H^m_l(\mathbb{R}_v^3)} < \infty.$$

Noting that

$$\partial_t \partial_v^\alpha f = \partial_v^\alpha Q(f,f) = Q(f,\partial_v^\alpha f) + \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ |\alpha_1|\geq 1}} C_{\alpha_1}^\alpha Q(\partial_v^{\alpha_1} f, \partial_v^{\alpha_2} f).$$

Set $g = \partial_v^\alpha f\langle v\rangle^l$, then g solves

$$\partial_t g = Q(f,g) + [Q(f,\partial_v^\alpha f)\langle v\rangle^l - Q(f,\partial_v^\alpha f\langle v\rangle^l)] + Q\sum_{\substack{\alpha_1+\alpha_2=\alpha \\ |\alpha_1|\geq 1}} C_{\alpha_1}^\alpha (\partial_v^{\alpha_1} f, \partial_v^{\alpha_2} f)\langle v\rangle^l.$$

Suppose $|\alpha| = m+1$. Thanks to the Theorem 3.1 and standard interpolation inequality, one has

$$\langle Q(f,g), g\rangle_v \lesssim -\frac{C_f}{2}\|g\langle v\rangle^{\frac{\gamma}{2}}\|^2_{H^s} + (C_f^{1-2s}\|f\|^{2s}_{L^2_{|\gamma|+4}} + \|f\|_{L^2_{|\gamma|+4}})\|f\|^2_{H^{m+\varrho+1}_{l+\frac{\gamma}{2}}},$$

with $\varrho < s$.

Due to Corollary 2.2, taking $N_2 = l + \frac{\gamma}{2}$ and $N_3 = \frac{\gamma}{2}$ in the case of $s < \frac{1}{2}$, otherwise taking $N_2 = l + \frac{\gamma}{2}$ and $N_3 = \frac{\gamma}{2} + 2s - 1$, one may arrive at

$$\langle Q(f, \partial_v^\alpha f) \langle v \rangle^l - Q(f, \partial_v^\alpha f \langle v \rangle^l), g \rangle_v$$
$$\lesssim \|f\|_{L^2_{|\gamma|+2s+2l+4}} (\|g \langle v \rangle^{\frac{\gamma}{2}}\|_{H^\varrho}^2 + \|g \langle v \rangle^{\frac{\gamma}{2}}\|_{H^s} \|f\|_{H^{m+1}_{l+\frac{\gamma}{2}+2s}}^2).$$

Applying the Theorem 2.1 with $N_2 = 2s + \frac{\gamma}{2} + l$ and $N_3 = -l + \frac{\gamma}{2}$, one may have

$$\langle Q(\partial_v^{\alpha_1} f, \partial_v^{\alpha_2} f), \langle v \rangle^l g \rangle_v \lesssim \|\partial_v^{\alpha_1} f\|_{L^2_{|\gamma|+2s+3l+4}} \|\partial_v^{\alpha_2} f\|_{H^s_{2s+\frac{\gamma}{2}+l}} \|g \langle v \rangle^{\frac{\gamma}{2}}\|_{H^s}.$$

Patching together all the above estimates, by standard energy estimate, we easily deduce that

$$\frac{d}{dt} \|g\|_{L^2}^2 + \frac{C_f}{3} \|g \langle v \rangle^{\frac{\gamma}{2}}\|_{H^s}^2 \lesssim \|f\|_{H^{m+1}_{l+\frac{\gamma}{2}+2s}}^2 + \|f\|_{H^{m+\varrho+1}_{l+\frac{\gamma}{2}}}^2,$$

where we use the Young's inequality and (4.61).

From which, one obtains that

$$(4.62) \qquad \frac{d}{dt} \|f\|_{H^{m+1}_l}^2 + \|f\|_{H^{m+1+s}_{l+\frac{\gamma}{2}}}^2 \lesssim \|f\|_{H^{m+1}_{l+\frac{\gamma}{2}+2s}}^2 + \|f\|_{H^{m+\varrho+1}_{l+\frac{\gamma}{2}}}^2.$$

Thanks to the interpolation inequality

$$\|f\|_{H^k_p}^2 \lesssim \|f\|_{H^{k-\epsilon}_{2p}} \|f\|_{H^{k+\epsilon}_0},$$

by iteration argument, one has that there exists a constant r_p and $\delta \in (0, 1)$ such that

$$(4.63) \qquad \|f\|_{H^k_p} \lesssim \|f\|_{H^{k-1}_{r_p}}^\delta \|f\|_{H^{k+\epsilon}_0}^{1-\delta}.$$

Denote $\mathfrak{C}_m$ as the quantity depending only on $\sup\limits_{[t_0, \infty)} \|f\|_{H^m_l(\mathbb{R}^3_v)}$ with $l \in \mathbb{R}^+$. Then by using (4.63), (4.62) can be rewritten as

$$\frac{d}{dt} \|f\|_{H^{m+1}_l}^2 + \|f\|_{H^{m+1+s}_{l+\frac{\gamma}{2}}}^2 \lesssim \mathfrak{C}_m.$$

Thanks to (4.63) again, we also can derive that there exists a constant $\eta > 0$ such that

$$\frac{d}{dt} \|f\|_{H^{m+1}_l}^2 + \|f\|_{H^{m+1}_l}^{2+\eta} \lesssim \mathfrak{C}_m.$$

Using a standard argument(used by Nash for parabolic equations), we see that for $t_1 > t_0$, there holds

$$f \in L^\infty([t_1, \infty]; H^{m+1}_l)$$

with $l \in \mathbb{R}^+$. This gives the proof to the Theorem 1.1.

Now we only need to check that infinite L^1 moment estimate (1.11) will imply infinite L^2 moment estimate (1.12) in the case of $\gamma + 2s > 0$. Set $\alpha = 0$, then $g = f \langle v \rangle^l$. Following the similar procedure, one may obtain that

$$\frac{d}{dt} \|g\|_{L^2}^2 + \frac{C_f}{3} \|g \langle v \rangle^{\frac{\gamma}{2}}\|_{H^s}^2$$
$$\lesssim (C_f^{1-2s} \|f\|_{L^1_{|\gamma|+2}}^{2s} + \|f\|_{L^1_{|\gamma|+2}}) \|g \langle v \rangle^{\frac{\gamma}{2}}\|_{H^\varrho}^2 + \|f\|_{L^1_{|\gamma|+2s+2l+2}} \|g \langle v \rangle^{\frac{\gamma}{2}}\|_{H^s} \|g\|_{L^2_{\frac{\gamma}{2}+2s}}.$$

Thanks to the interpolation inequality

$$\|f\|_{H^\varrho} \lesssim \|f\|_{L^1}^{\frac{2s-2\varrho}{2s+3}} \|f\|_{H^s}^{\frac{3+2\varrho}{2s+3}}$$

and

$$\|f\|_{L^2} \leq \|f \langle v \rangle^{\frac{-3r}{2s+3}}\|_{L^1}^{\frac{2s}{2s+3}} \|f \langle v \rangle^r\|_{H^s}^{\frac{3}{2s+3}}$$

with $r \in \mathbb{R}$, one may deduce that there exists a constant $\eta > 0$ such that

$$\frac{d}{dt}\|f\langle v\rangle^l\|_{L^2}^2 + \|f\langle v\rangle^l\|_{L^2}^{2+\eta} \lesssim \mathfrak{C}_0^1,$$

where $\mathfrak{C}_0^1$ represents the quantity depending only on $\sup_{[t_0,\infty)} \|f\langle v\rangle^l\|_{L^1(\mathbb{R}_v^3)}$ with $l \in \mathbb{R}^+$. From which, we complete the proof to the Theorem 1.1. $\blacksquare$

5. SMOOTHING EFFECT FOR THE INHOMOGENEOUS BOLTZMANN EQUATION

5.1. Hypoelliptic estimate for the transport equation.
In this section we study the transport equation which reads:

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = g(t,x,v) \tag{5.64}$$

and show the following hypoelliptic estimate.

Lemma 5.1. *Suppose $g \in L^2([0,T] \times \mathbb{T}^3; H^{-1}(\mathbb{R}_v^3))$. Let $f \in L^2([0,T] \times \mathbb{T}^3 \times \mathbb{R}^3)$ be a weak solution of the transport equation* (5.64). *If we assume $f(0,x,v) \in L^2(\mathbb{T}^3 \times \mathbb{R}^3)$ and $f \in L^2([0,T] \times \mathbb{T}^3; \dot{H}^s(\mathbb{R}_v^3))$ for some $0 < s < 1$, then we have that for any $l < -\frac{3}{2}$,*

$$\langle v\rangle^l f \in L^2([0,T] \times \mathbb{R}^3; \dot{H}^{\frac{s}{4(4+s)}}(\mathbb{T}_x^3)). \tag{5.65}$$

Proof: Let τ_k be the translation operator in the x variable by k, then one has

$$\tau_k f(t,x,v) = f(t,x+k,v) - f(t,x,v).$$

We denote the finite difference of f in the x variable by

$$\triangle_k f(t,x,v) = \tau_k f(t,x,v) - f(t,x,v).$$

Using these notations, we observe that

$$(5.65) \;\Leftrightarrow\; \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{T}^3} \int_{\mathbb{T}^3} \langle v\rangle^{2l} |\triangle_k f|^2 |k|^{-3-\frac{s}{2(4+s)}} \, dt\,dv\,dx\,dk < +\infty$$

$$(5.66) \qquad \Leftrightarrow\; \int_0^T \int_{\mathbb{R}^3} \langle v\rangle^{2l} \left(\sum_{m \in \mathbb{Z}^3} |m|^{\frac{s}{2(4+s)}} \left| \hat{f}(t,m,v) \right|^2 \right) dt\,dv < +\infty.$$

We now turn to prove (5.66). Let $\chi(v) \in C_0^\infty(\mathbb{R}^3)$ be a test function which satisfies $\chi(v) \geq 0$ and $\int_{\mathbb{R}^3} \chi(v)dv = 1$. For any $\epsilon > 0$, we denote the regularizing sequence χ_ϵ by $\chi_\epsilon(v) = \epsilon^{-3}\chi\left(\frac{v}{\epsilon}\right)$ and write

$$\hat{f}(t,m,v) = [\hat{f}(t,m,v) - (\hat{f}(t,m,\cdot) *_v \chi_\epsilon)(v)] + (\hat{f}(t,m,\cdot) *_v \chi_\epsilon)(v). \tag{5.67}$$

We point out here that ϵ in the above equality will be chosen later and will depend on $|m|$.

We use Minkowski's inequality and Cauchy-Schwartz inequality to get

$$\int_{\mathbb{R}^3} \langle v\rangle^{2l} |\hat{f}(t,m,v) - (\hat{f}(t,m,\cdot) *_v \chi_\epsilon)(v)|^2 dv$$

$$\lesssim \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} [\hat{f}(t,m,v) - \hat{f}(t,m,v-u)] \chi_\epsilon(u)du \right|^2 dv$$

$$\lesssim \left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |\hat{f}(t,m,v) - \hat{f}(t,m,v-u)|^2 dv \right)^{1/2} \chi_\epsilon(u)du \right)^2$$

$$\lesssim \left(\int_{\mathbb{R}^3} \chi_\epsilon^2(u)|u|^{3+2s} du \right) \left(\int_{\mathbb{R}^6} \frac{|\hat{f}(t,m,v) - \hat{f}(t,m,v-u)|^2}{|u|^{3+2s}} du\,dv \right)$$

$$\lesssim \epsilon^{2s} \left\| \hat{f}(t,m,\cdot) \right\|_{\dot{H}^s(\mathbb{R}_v^3)}^2,$$

and we then obtain

$$\int_0^T \int_{\mathbb{R}^3} \langle v \rangle^{2l} \left(\sum_{m \in \mathbb{Z}^3} |m|^{\frac{s}{2(4+s)}} \left| \hat{f}(t,m,v) - (\hat{f}(t,m,\cdot) *_v \chi_\epsilon)(v) \right|^2 \right) dt dv$$

$$(5.68) \qquad \lesssim \int_0^T \left(\sum_{m \in \mathbb{Z}^3} |m|^{\frac{s}{2(4+s)}} \epsilon^{2s} \left\| \hat{f}(t,m,\cdot) \right\|_{\dot{H}^s(\mathbb{R}_v^3)}^2 \right) dt.$$

For the second term of the right-hand side of (5.67), we shall use the averaging lemma introduced by [19]. We first recall that $g \in L^2([0,T] \times \mathbb{T}^3; H^{-1}(\mathbb{R}_v^3))$ implies that

$$g(t,x,v) = g_0(t,x,v) + \sum_{j=1}^3 \partial_{v_j} h_j(t,x,v),$$

where $g_0(t,x,v) = \mathcal{F}_v^{-1}[(1+|\xi|)^{-1}\mathcal{F}_v g](t,x,v)$ and $h_j(t,x,v) = -R_j g_0(t,x,v)$, $j = 1,2,3$. Here R_j is Riesz transform in v variable. Then, one has $g_0, h_j \in L^2([0,T] \times \mathbb{T}^3 \times \mathbb{R}^3)$, $j = 1,2,3$. According to (2.16) in Theorem 2.1 (averaging lemma) of [19], we can deduce

$$\int_0^T |(\hat{f}(t,m,\cdot) *_v \chi_\epsilon)(v)|^2 dt$$

$$\lesssim |m|^{-\frac{1}{2}} \left(\|\chi_\epsilon(v-u)(1+|u|^2)\|_{L_u^\infty} + \|\nabla \chi_\epsilon(v-u)(1+|u|^2)\|_{L_u^\infty} \right)^2$$

$$\times \left(\|\hat{f}(0,m,\cdot)\|_{L^2(\mathbb{R}_v^3)}^2 + \|\hat{f}(\cdot,m,\cdot)\|_{L^2([0,T];L^2(\mathbb{R}_v^3))}^2 \right.$$

$$\left. + \|\hat{g}_0(\cdot,m,\cdot)\|_{L^2([0,T];L^2(\mathbb{R}_v^3))}^2 + \sum_{j=1}^3 \|\hat{h}_j(\cdot,m,\cdot)\|_{L^2([0,T];L^2(\mathbb{R}_v^3))}^2 \right).$$

Thanks to the fact $\|\chi_\epsilon(v-u)(1+|u|^2)\|_{L_u^\infty} \lesssim \epsilon^{-3}(1+|v|^2)$, we get

$$\int_0^T \int_{\mathbb{R}^3} \langle v \rangle^{2l} \left(\sum_{m \in \mathbb{Z}^3} |m|^{\frac{s}{2(4+s)}} \left| (\hat{f}(t,m,\cdot) *_v \chi_\epsilon)(v) \right|^2 \right) dt dv$$

$$\lesssim \sum_{m \in \mathbb{Z}^3} |m|^{\frac{s}{2(4+s)}-\frac{1}{2}}(\epsilon^{-6} + \epsilon^{-8}) \left(\|\hat{f}(0,m,\cdot)\|_{L^2(\mathbb{R}_v^3)}^2 + \|\hat{f}(\cdot,m,\cdot)\|_{L^2([0,T];L^2(\mathbb{R}_v^3))}^2 \right.$$

$$(5.69) \qquad \left. + \|\hat{g}_0(\cdot,m,\cdot)\|_{L^2([0,T];L^2(\mathbb{R}_v^3))}^2 + \sum_{j=1}^3 \|\hat{h}_j(\cdot,m,\cdot)\|_{L^2([0,T];L^2(\mathbb{R}_v^3))}^2 \right).$$

Now we choose $\epsilon = |m|^{-\frac{1}{4(4+s)}}$, and we can bound (5.68) since we have $f \in L^2([0,T] \times \mathbb{T}^3; \dot{H}^s(\mathbb{R}_v^3))$. As for (5.69), we can also bound it if we notice the fact that $f(0,x,v) \in L^2(\mathbb{T}^3 \times \mathbb{R}^3)$ and $g_0, h \in L^2([0,T] \times \mathbb{T}^3 \times \mathbb{R}^3)$. This completes the proof of Lemma 5.1. ∎

Remark 5.1. *The idea of the proof of Lemma 5.1 has been used in that of Lemma 4.2 of [24] which is devoted to the smoothing effects for classical solutions of the full Landau equation. We point out that we actually proved Lemma 5.1 in the case $s = 1$ there.*

Now we start to prove Theorem 1.2 that gives the smoothing effect for the inhomogeneous Boltzmann equation (1.1). We first note that if no confuse occurs, we omit the domains $\mathbb{T}^3$ and $\mathbb{R}^3$, which correspond to variables x and v respectively for simplicity, and we use the shorthand $\partial_\beta^\alpha = \partial_x^\alpha \partial_v^\beta$ for any multi-indices α and β hereafter.

In order to prove our main result, we shall use an induction on the number of derivatives (in variables x and v) that can be controlled. One step of this induction is given by

Proposition 5.1. *Let $N \geq 5$ be a given integer, and let f be a smooth solution to the Boltzmann equation (1.1). For any $l \geq 0$, we set $h = (\partial_\beta^\alpha f)\langle v \rangle^l$ with $|\alpha| + |\beta| \leq N$. We assume that for any $T > 0$ and any $l \geq 0$, $h \in L_t^\infty([0,T]; L_{x,v}^2)$. Then we have that $h \in L_t^\infty([\tau,T]; H_{x,v}^1)$ for any time $\tau \in (0,T)$.*

We only prove the above proposition in the case $\gamma + 2s > 0$, the proofs for the case $\gamma + 2s \leq 0$ are analogous if we use the estimates of this case in Theorem 2.1, Corollary 2.1 and Theorem 3.1 instead of those of the case $\gamma + 2s > 0$. We start the proof with improving regularity in x variable.

5.2. Gain regularity in x variable.
We note the domain of variable t should be $[0,T]$ if we omit it hereafter. We present following

Lemma 5.2. *Let $N \geq 5$ be a given integer, and let f be a smooth solution to the Boltzmann equation (1.1). We suppose that for any $T > 0$ and any $l \geq 0$, $h \in L_{t,x,v}^2$ and $h(0) \in L_{x,v}^2$, where h is defined in Proposition 5.1. We further suppose that $(\partial_\beta^\alpha f)\langle v \rangle^l \in L_t^\infty(L_{x,v}^2)$ for $|\alpha| + |\beta| \leq N - 1$. Then we have $h \in L_t^\infty(L_{x,v}^2) \cap L_{t,x}^2(H_v^s)$.*

Proof: Using Einstein's convention for repeated indices, we have that h satisfies the equation as follows: for $|\beta| = 0$,

$$(5.70) \qquad \partial_t h + v \cdot \nabla_x h = [\partial_x^\alpha Q(f,f)]\langle v \rangle^l,$$

and for $|\beta| \geq 1$,

$$(5.71) \qquad \partial_t h + v \cdot \nabla_x h = -\beta_i(\partial_{\beta-e_i}^{\alpha+e_i} f)\langle v \rangle^l + [\partial_\beta^\alpha Q(f,f)]\langle v \rangle^l,$$

where $e_1 = (1,0,0), e_2 = (0,1,0)$ and $e_3 = (0,0,1)$.

We only consider the case $|\beta| \geq 1$, because the estimates for the case $|\beta| = 0$ are similar (and easier). Multiplying equation (5.71) by h, and then integrating on (t,x,v), we shall estimate the resulting equation term by term.

It is easy to see that

$$(5.72) \qquad \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} (\partial_t h) h \, dt dx dv = \frac{1}{2}\left(\|h(T)\|_{L_{x,v}^2}^2 - \|h(0)\|_{L_{x,v}^2}^2 \right).$$

Since f is a spatially periodic function, we get that

$$(5.73) \qquad \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} (v \cdot \nabla_x h) h \, dt dx dv = \frac{1}{2}\int_0^T \int_{\mathbb{R}^3} v \cdot \left(\int_{\mathbb{T}^3} \nabla_x(h^2) dx \right) dt dv = 0.$$

Cauchy-Schwartz gives

$$(5.74) \qquad \left| \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} -\beta_i(\partial_{\beta-e_i}^{\alpha+e_i} f)\langle v \rangle^l h \, dt dx dv \right| \lesssim \|(\partial_{\beta-e_i}^{\alpha+e_i} f)\langle v \rangle^l\|_{L_{t,x,v}^2} \|h\|_{L_{t,x,v}^2}.$$

We write

$$\int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} [\partial_\beta^\alpha Q(f,f)]\langle v \rangle^l h \, dt dx dv = \int_0^T \int_{\mathbb{T}^3} \langle [\partial_\beta^\alpha Q(f,f)]\langle v \rangle^l, h \rangle_v dt dx$$

$$= \int_0^T \int_{\mathbb{T}^3} \langle Q(f,h), h \rangle_v dt dx + \int_0^T \int_{\mathbb{T}^3} \langle Q(f, \partial_\beta^\alpha f)\langle v \rangle^l - Q(f,h), h \rangle_v dt dx$$

$$+ \int_0^T \int_{\mathbb{T}^3} \sum_{\substack{\alpha_1 + \alpha_2 = \alpha \\ \beta_1 + \beta_2 = \beta \\ |\alpha_1| + |\beta_1| \geq 1}} C_{\alpha_1}^\alpha C_{\beta_1}^\beta \langle Q(\partial_{\beta_1}^{\alpha_1} f, (\partial_{\beta_2}^{\alpha_2} f)\langle v \rangle^l), h \rangle_v dt dx$$

$$+ \int_0^T \int_{\mathbb{T}^3} \sum_{\substack{\alpha_1 + \alpha_2 = \alpha \\ \beta_1 + \beta_2 = \beta \\ |\alpha_1| + |\beta_1| \geq 1}} C_{\alpha_1}^\alpha C_{\beta_1}^\beta \langle Q(\partial_{\beta_1}^{\alpha_1} f, \partial_{\beta_2}^{\alpha_2} f)\langle v \rangle^l - Q(\partial_{\beta_1}^{\alpha_1} f, (\partial_{\beta_2}^{\alpha_2} f)\langle v \rangle^l), h \rangle_v dt dx$$

$$(5.75) \overset{\text{def}}{=} (J_1) + (J_2) + (J_3) + (J_4).$$

Theorem 3.1 gives

$$\langle Q(f,h),h\rangle_v \;\lesssim\; -C_f\|h\langle v\rangle^{\frac{\gamma}{2}}\|_{H^s_v} + (C_f^{1-2s}\|f\langle v\rangle^{\tilde\gamma}\|^{2s}_{L^1_v} + \|f\langle v\rangle^{\tilde\gamma}\|_{L^1_v})\|h\langle v\rangle^{\frac{\gamma}{2}}\|^2_{L^2_v}$$
$$+\|f\langle v\rangle^{|\gamma|}\|_{L^1_v}\|h\langle v\rangle^{\frac{\gamma}{2}}\|^2_{H^\varrho_v},$$

where $\varrho < s$. We remark here that the constant C_f is uniformly with respect to x variables due to the Proposition 3 of [5] and the assumption (1.15). It is easy to get by interpolation and Young's inequality that

$$(5.76) \qquad\qquad \|h\langle v\rangle^{\frac{\gamma}{2}}\|^2_{H^\varrho_v} \le \epsilon\|h\langle v\rangle^{\frac{\gamma}{2}}\|^2_{H^s_v} + C_\epsilon\|h\langle v\rangle^{\frac{\gamma}{2}}\|^2_{L^2_v},$$

which implies

$$(J_1) \;\lesssim\; -(C_f - \epsilon\|f\langle v\rangle^{|\gamma|+2}\|_{L^\infty_t(H^2_x(L^2_v))})\|h\langle v\rangle^{\frac{\gamma}{2}}\|^2_{L^2_{t,x}(H^s_v)} + (C_f^{1-2s}\|f\langle v\rangle^{\tilde\gamma+2}\|^{2s}_{L^\infty_t(H^2_x(L^2_v))}$$
$$(5.77) \qquad + \|f\langle v\rangle^{\tilde\gamma+2}\|_{L^\infty_t(H^2_x(L^2_v))} + C_\epsilon\|f\langle v\rangle^{|\gamma|+2}\|_{L^\infty_t(H^2_x(L^2_v))})\|h\langle v\rangle^{\frac{\gamma}{2}}\|^2_{L^2_{t,x,v}},$$

where we use Sobolev's embedding theorem and the inequality $\|g\|_{L^1_v} \lesssim \|g\langle v\rangle^2\|_{L^2_v}$. As for the term (J_2), in the case $s \ge \frac{1}{2}$, Corollary 2.1 implies

$$|(J_2)| \lesssim \int_0^T \int_{\mathbb{T}^3} \|f\langle v\rangle^{N_1}\|_{L^1_v}\|(\partial^\alpha_\beta f)\langle v\rangle^{N_2}\|_{H^s_v}\|h\langle v\rangle^{N_3}\|_{L^2_v}\,dtdx.$$

If we choose $N_1 = |l+\frac{\gamma}{2}| + |\frac{\gamma}{2}+2s-1| + \max\{|l-2|,|l-1|\}$, $N_2 = l+\frac{\gamma}{2}$ and $N_3 = \frac{\gamma}{2}+2s-1$, we get by Sobolev's embedding theorem and Young's inequality that

$$(5.78) \qquad |(J_2)| \lesssim \epsilon\|h\langle v\rangle^{\frac{\gamma}{2}}\|^2_{L^2_{t,x}(H^s_v)} + C_\epsilon\|f\langle v\rangle^{N_1+2}\|^2_{L^\infty_t(H^2_x(L^2_v))}\|h\langle v\rangle^{N_3}\|^2_{L^2_{t,x,v}}.$$

In the case $s < \frac{1}{2}$, we again use Corollary 2.1 to get

$$|(J_2)| \lesssim \int_0^T \int_{\mathbb{T}^3} \|f\langle v\rangle^{N_1}\|_{L^1_v}\|(\partial^\alpha_\beta f)\langle v\rangle^{N_2}\|_{H^\rho_v}\|h\langle v\rangle^{N_3}\|_{H^\rho_v}\,dtdx,$$

where $\rho < s$. Taking $N_1 = |l+\frac{\gamma}{2}| + |\frac{\gamma}{2}| + \max\{|l-2|,|l-1|\}$, $N_2 = l+\frac{\gamma}{2}$ and $N_3 = \frac{\gamma}{2}$, we obtain by using (5.76) that
(5.79)
$$|(J_2)| \lesssim \epsilon\|f\langle v\rangle^{N_1+2}\|_{L^\infty_t(H^2_x(L^2_v))}\|h\langle v\rangle^{\frac{\gamma}{2}}\|^2_{L^2_{t,x}(H^s_v)} + C_\epsilon\|f\langle v\rangle^{N_1+2}\|^2_{L^\infty_t(H^2_x(L^2_v))}\|h\langle v\rangle^{\frac{\gamma}{2}}\|^2_{L^2_{t,x,v}}.$$

To deal with the term (J_3), we shall consider two cases. In the case $1 \le |\alpha_1| + |\beta_1| \le \left[\frac{N}{2}\right]$ (for $r \in \mathbb{R}, [r]$ denotes the maximum integer which is less than or equal to r), we have $|\alpha_1| + |\beta_1| + 2 \le N - 1$ and $|\alpha_2| + |\beta_2| \le N - 1$. Then Theorem 2.1 gives

$$\left|\int_0^T \int_{\mathbb{T}^3} \langle Q(\partial^{\alpha_1}_{\beta_1} f, (\partial^{\alpha_2}_{\beta_2} f)\langle v\rangle^l), h\rangle_v\,dtdx\right|$$
$$\lesssim \|(\partial^{\alpha_1}_{\beta_1} f)\langle v\rangle^{N_1+2}\|_{L^\infty_{t,x}(L^2_v)}\|(\partial^{\alpha_2}_{\beta_2} f)\langle v\rangle^{l+N_2}\|_{L^2_{t,x}(H^s_v)}\|h\langle v\rangle^{N_3}\|_{L^2_{t,x}(H^s_v)},$$

where we take $N_1 = |\frac{\gamma}{2}+2s| + |\frac{\gamma}{2}|$, $N_2 = \frac{\gamma}{2}+2s$, $N_3 = \frac{\gamma}{2}$. And Young's inequality gives

$$(5.80) \quad |(J_3)| \lesssim \epsilon\|h\langle v\rangle^{\frac{\gamma}{2}}\|^2_{L^2_{t,x}(H^s_v)} + C_\epsilon\|(\partial^{\alpha_1}_{\beta_1} f)\langle v\rangle^{N_1+2}\|^2_{L^\infty_t(H^2_x(L^2_v))}\|(\partial^{\alpha_2}_{\beta_2} f)\langle v\rangle^{l+N_2}\|^2_{L^2_{t,x}(H^s_v)}$$

In the case $|\alpha_1| + |\beta_1| \ge \left[\frac{N}{2}\right] + 1$, we see $|\alpha_2| + |\beta_2| + 2 + s \le N - 1$. Again by Theorem 2.1,

$$\left|\int_0^T \int_{\mathbb{T}^3} \langle Q(\partial^{\alpha_1}_{\beta_1} f, (\partial^{\alpha_2}_{\beta_2} f)\langle v\rangle^l), h\rangle_v\,dtdx\right|$$
$$\lesssim \|(\partial^{\alpha_1}_{\beta_1} f)\langle v\rangle^{N_1+2}\|_{L^2_{t,x}(L^2_v)}\|(\partial^{\alpha_2}_{\beta_2} f)\langle v\rangle^{l+N_2}\|_{L^\infty_{t,x}(H^s_v)}\|h\langle v\rangle^{N_3}\|_{L^2_{t,x}(H^s_v)},$$

where we take $N_1 = |\frac{\gamma}{2}+2s| + |\frac{\gamma}{2}|$, $N_2 = \frac{\gamma}{2}+2s$, $N_3 = \frac{\gamma}{2}$. And Young's inequality gives

$$(5.81) \quad |(J_3)| \lesssim \epsilon\|h\langle v\rangle^{\frac{\gamma}{2}}\|^2_{L^2_{t,x}(H^s_v)} + C_\epsilon\|(\partial^{\alpha_1}_{\beta_1} f)\langle v\rangle^{N_1+2}\|^2_{L^2_{t,x,v}}\|(\partial^{\alpha_2}_{\beta_2} f)\langle v\rangle^{l+N_2}\|^2_{L^\infty_t(H^2_x(H^s_v))}.$$

Similarly, using Corollary 2.1, we can bound (J_4) by the quantities which have been controlled and $\epsilon\|h\langle v\rangle^{\frac{\gamma}{2}}\|^2_{L^2_{t,x}(H^s_v)}$ if we study the case $1 \leq |\alpha_1| + |\beta_1| \leq \left[\frac{N}{2}\right]$ and the case $|\alpha_1| + |\beta_1| \geq \left[\frac{N}{2}\right] + 1$ respectively.

If we choose ϵ appeared small enough and collect all the above estimates, we arrive at $h \in L^\infty_t(L^2_{x,v}) \cap L^2_{t,x}(H^s_v)$. This ends up the proof. ∎

Now we begin to improve the regularity in x variable. We have

Lemma 5.3. *Let $N \geq 5$ be a given integer, and let f be a smooth solution to the Boltzmann equation (1.1). We suppose that for any $T > 0$ and any $l \geq 0$, $h \in L^\infty_t(L^2_{x,v})$ and $h(0) \in L^2_{x,v}$, where h is defined in Proposition 5.1. Then we have $h \in L^2_{t,v}(\dot{H}^{\frac{s}{4(4+s)}}_x)$.*

Proof: We still only consider the case $|\beta| \geq 1$. Recall that h satisfies the equation (5.71), then we get

$$\partial_t h + v \cdot \nabla_x h = -\beta_i(\partial^{\alpha+e_i}_{\beta-e_i} f)\langle v\rangle^l + \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} C^\alpha_{\alpha_1} C^\beta_{\beta_1} Q(\partial^{\alpha_1}_{\beta_1} f, (\partial^{\alpha_2}_{\beta_2} f)\langle v\rangle^l)$$

$$+ \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} C^\alpha_{\alpha_1} C^\beta_{\beta_1} \left[Q(\partial^{\alpha_1}_{\beta_1} f, \partial^{\alpha_2}_{\beta_2} f)\langle v\rangle^l - Q(\partial^{\alpha_1}_{\beta_1} f, (\partial^{\alpha_2}_{\beta_2} f)\langle v\rangle^l) \right]$$

(5.82) $$\stackrel{\text{def}}{=} (K_1) + (K_2) + (K_3).$$

It is obvious that $(K_1) \in L^2_{t,x,v}$. Theorem 2.1 gives (taking $N_1 = N_2 = \gamma + 2s$, $N_3 = 0$)

$$\|Q(\partial^{\alpha_1}_{\beta_1} f, (\partial^{\alpha_2}_{\beta_2} f)\langle v\rangle^l)\|_{H^{-s}_v} \lesssim \|(\partial^{\alpha_1}_{\beta_1} f)\langle v\rangle^{\gamma+2s}\|_{L^1_v}\|(\partial^{\alpha_2}_{\beta_2} f)\langle v\rangle^{\gamma+2s+l}\|_{H^s_v}$$

$$\lesssim \|(\partial^{\alpha_1}_{\beta_1} f)\langle v\rangle^{\gamma+2s+2}\|_{L^2_v}\|(\partial^{\alpha_2}_{\beta_2} f)\langle v\rangle^{\gamma+2s+l}\|_{H^s_v}.$$

Then in the case $|\alpha_1| + |\beta_1| \leq \left[\frac{N}{2}\right]$, one has $|\alpha_1| + |\beta_1| + 2 \leq N - 1$, so that

$$\|Q(\partial^{\alpha_1}_{\beta_1} f, (\partial^{\alpha_2}_{\beta_2} f)\langle v\rangle^l)\|_{L^2_{t,x}(H^{-s}_v)} \lesssim \|(\partial^{\alpha_1}_{\beta_1} f)\langle v\rangle^{\gamma+2s+2}\|_{L^\infty_t(H^2_x(L^2_v))}\|(\partial^{\alpha_2}_{\beta_2} f)\langle v\rangle^{\gamma+2s+l}\|_{L^2_{t,x}(H^s_v)},$$

while in the case $|\alpha_1| + |\beta_1| \geq \left[\frac{N}{2}\right] + 1$, one has $|\alpha_2| + |\beta_2| + 2 + s \leq N - 1$, so that

$$\|Q(\partial^{\alpha_1}_{\beta_1} f, (\partial^{\alpha_2}_{\beta_2} f)\langle v\rangle^l)\|_{L^2_{t,x}(H^{-s}_v)} \lesssim \|(\partial^{\alpha_1}_{\beta_1} f)\langle v\rangle^{\gamma+2s+2}\|_{L^2_{t,x,v}}\|(\partial^{\alpha_2}_{\beta_2} f)\langle v\rangle^{\gamma+2s+l}\|_{L^\infty_t(H^2_x(H^s_v))}.$$

We thus obtain that $(K_2) \in L^2_{t,x}(H^{-s}_v)$ by Lemma 5.2. In the case $s \geq \frac{1}{2}$, Corollary 2.1 gives (taking $N_1 = |l - 1 + \gamma + 2s| + \max\{|l - 2|, |l - 1|\}$, $N_2 = l - 1 + \gamma + 2s$ and $N_3 = 0$)

$$\|Q(\partial^{\alpha_1}_{\beta_1} f, \partial^{\alpha_2}_{\beta_2} f)\langle v\rangle^l - Q(\partial^{\alpha_1}_{\beta_1} f, (\partial^{\alpha_2}_{\beta_2} f)\langle v\rangle^l)\|_{L^2_v}$$

$$\lesssim \|(\partial^{\alpha_1}_{\beta_1} f)\langle v\rangle^{N_1+2}\|_{L^2_v}\|(\partial^{\alpha_2}_{\beta_2} f)\langle v\rangle^{N_2}\|_{H^s_v}.$$

Considering the case $|\alpha_1| + |\beta_1| \leq \left[\frac{N}{2}\right]$ and the case $|\alpha_1| + |\beta_1| \geq \left[\frac{N}{2}\right] + 1$ respectively just as the estimates for (K_2), we can deduce that $(K_3) \in L^2_{t,x,v}$. While in the case $s < \frac{1}{2}$, again by Corollary 2.1, we get for $\rho \in (0, s)$,

$$\|Q(\partial^{\alpha_1}_{\beta_1} f, \partial^{\alpha_2}_{\beta_2} f)\langle v\rangle^l - Q(\partial^{\alpha_1}_{\beta_1} f, (\partial^{\alpha_2}_{\beta_2} f)\langle v\rangle^l)\|_{H^{-\rho}_v}$$

$$\lesssim \|(\partial^{\alpha_1}_{\beta_1} f)\langle v\rangle^{N_1+2}\|_{L^2_v}\|(\partial^{\alpha_2}_{\beta_2} f)\langle v\rangle^{N_2}\|_{H^\rho_v}$$

where we take $N_1 = |l + \gamma| + \max\{|l - 2|, |l - 1|\}$, $N_2 = l + \gamma$ and $N_3 = 0$. Still considering the case $|\alpha_1| + |\beta_1| \leq \left[\frac{N}{2}\right]$ and the case $|\alpha_1| + |\beta_1| \geq \left[\frac{N}{2}\right] + 1$ respectively, we may obtain that $(K_3) \in L^2_{t,x}(H^{-\rho}_v) \subset L^2_{t,x}(H^{-s}_v)$.

Now we have that $(K_1)+(K_2)+(K_3) \in L^2_{t,x}(H^{-s}_v) \subset L^2_{t,x}(H^{-1}_v)$, and hence $h \in L^2_{t,v}(\dot{H}^{\frac{s}{4(4+s)}}_x)$ if we use Lemma 5.1. ∎

Roughly speaking, Lemma 5.3 shows that if $h \in L_t^\infty(L_{x,v}^2)$, then h has a $\frac{s}{4(4+s)}$ derivative gain in x variable. The next step is to prove that h can still gain a $\frac{s}{4(4+s)}$ derivative in x variable, so that we can finally reach that $h \in L_t^\infty(H_x^1(L_v^2))$ if we repeat this step several times. To this end, we present following

Lemma 5.4. *We denote $\delta = \frac{s}{4(4+s)}$ for simplicity. Let $N \geq 5$ be a given integer, and let f be a smooth solution to the Boltzmann equation (1.1). We suppose that for any $T > 0$ and any $l \geq 0$, $h \in L_t^\infty(L_{x,v}^2) \cap L_{t,v}^2(\dot{H}_x^\delta)$ and $h(0) \in L_v^2(\dot{H}_x^\delta)$ where h is defined in Proposition 5.1. Then we have $g_{\delta,k} \in L_{t,x,k}^2(H_v^s)$ with $g_{\delta,k} = \triangle_k h|k|^{-\delta - \frac{3}{2}}$, $k \in \mathbb{T}^3$.*

Proof: We still only consider the case $|\beta| \geq 1$. The equation for $g_{\delta,k}$ in this case reads:

$$
\begin{aligned}
&\partial_t g_{\delta,k} + v \cdot \nabla_x g_{\delta,k} \\
&= -\beta_i \left[\frac{\triangle_k(\partial_{\beta-e_i}^{\alpha+e_i} f)}{|k|^{\delta+\frac{3}{2}}} \right] \langle v \rangle^l + Q(f(x+k), g_{\delta,k}) \\
&\quad + \left[Q\left(f(x+k), \frac{\triangle_k(\partial_\beta^\alpha f)}{|k|^{\delta+\frac{3}{2}}} \right) \langle v \rangle^l - Q(f(x+k), g_{\delta,k}) \right] \\
&\quad + \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta \\ |\alpha_1|+|\beta_1| \geq 1}} C_{\alpha_1}^\alpha C_{\beta_1}^\beta Q\left(\partial_{\beta_1}^{\alpha_1} f(x+k), \frac{\triangle_k((\partial_{\beta_2}^{\alpha_2} f)(x))\langle v \rangle^l}{|k|^{\delta+\frac{3}{2}}} \right) \\
&\quad + \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta \\ |\alpha_1|+|\beta_1| \geq 1}} C_{\alpha_1}^\alpha C_{\beta_1}^\beta \left[Q\left(\partial_{\beta_1}^{\alpha_1} f(x+k), \frac{\triangle_k(\partial_{\beta_2}^{\alpha_2} f(x))}{|k|^{\delta+\frac{3}{2}}} \right) \langle v \rangle^l \right. \\
&\quad \left. -Q\left(\partial_{\beta_1}^{\alpha_1} f(x+k), \frac{\triangle_k((\partial_{\beta_2}^{\alpha_2} f)(x))\langle v \rangle^l}{|k|^{\delta+\frac{3}{2}}} \right) \right] \\
&\quad + \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} C_{\alpha_1}^\alpha C_{\beta_1}^\beta Q\left(\frac{\triangle_k(\partial_{\beta_1}^{\alpha_1} f)}{|k|^{\delta+\frac{3}{2}}}, (\partial_{\beta_2}^{\alpha_2} f)\langle v \rangle^l \right) \\
&\quad + \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} C_{\alpha_1}^\alpha C_{\beta_1}^\beta \left[Q\left(\frac{\triangle_k(\partial_{\beta_1}^{\alpha_1} f)}{|k|^{\delta+\frac{3}{2}}}, \partial_{\beta_2}^{\alpha_2} f \right) \langle v \rangle^l - Q\left(\frac{\triangle_k(\partial_{\beta_1}^{\alpha_1} f)}{|k|^{\delta+\frac{3}{2}}}, (\partial_{\beta_2}^{\alpha_2} f)\langle v \rangle^l \right) \right]
\end{aligned}
$$

$$(5.83) \overset{\text{def}}{=} (L_1) + (L_2) + (L_3) + (L_4) + (L_5) + (L_6) + (L_7).$$

Multiplying the above equation by $g_{\delta,k}$, then integrating on (t, x, v, k) in the domain $[0, T] \times \mathbb{T}^3 \times \mathbb{R}^3 \times \mathbb{T}^3$. Similar to the estimates (5.72)-(5.74), we can get

$$(5.84) \qquad \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{T}^3} (\partial_t g_{\delta,k}) g_{\delta,k} \, dt \, dx \, dv \, dk = \frac{1}{2} \left(\|g_{\delta,k}(T)\|_{L_{x,v,k}^2}^2 - \|g_{\delta,k}(0)\|_{L_{x,v,k}^2}^2 \right),$$

$$(5.85)$$
$$\int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{T}^3} (v \cdot \nabla_x g_{\delta,k}) g_{\delta,k} \, dt \, dx \, dv \, dk = \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{T}^3} v \cdot \left(\int_{\mathbb{T}^3} \nabla_x(g_{\delta,k}^2) \, dx \right) dt \, dv \, dk = 0$$

and

$$(5.86) \qquad \left| \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{T}^3} (L_1) g_{\delta,k} \, dt \, dx \, dv \, dk \right| \lesssim \|(\partial_{\beta-e_i}^{\alpha+e_i} f)\langle v \rangle^l\|_{L_{t,v}^2(\dot{H}_x^\delta)} \|g_{\delta,k}\|_{L_{t,x,v,k}^2}.$$

If we do the estimates like those for (J_1) and (J_2) in (5.75), we can use Theorem 3.1 and Corollary 2.1 to get

$$\left|\int_0^T \int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\int_{\mathbb{T}^3}(L_2)g_{\delta,k}dtdxdvdk\right|$$
$$\lesssim -(C_f - \epsilon\|f\langle v\rangle^{|\gamma|+2}\|_{L_t^\infty(H_x^2(L_v^2))})\|g_{\delta,k}\langle v\rangle^{\frac{\gamma}{2}}\|_{L_{t,x,k}^2(H_v^s)}^2 + (C_f^{1-2s}\|f\langle v\rangle^{\tilde\gamma+2}\|_{L_t^\infty(H_x^2(L_v^2))}^{2s}$$
$$(5.87)\quad +\|f\langle v\rangle^{\tilde\gamma+2}\|_{L_t^\infty(H_x^2(L_v^2))} + C_\epsilon\|f\langle v\rangle^{|\gamma|+2}\|_{L_t^\infty(H_x^2(L_v^2))})\|g_{\delta,k}\langle v\rangle^{\frac{\gamma}{2}}\|_{L_{t,x,v,k}^2}^2$$

and in the case $s \geq \frac{1}{2}$ (taking $N_1 = |l + \frac{\gamma}{2}| + |\frac{\gamma}{2} + 2s - 1| + \max\{|l-2|,|l-1|\}$, $N_2 = l + \frac{\gamma}{2}$ and $N_3 = \frac{\gamma}{2} + 2s - 1$),

$$\left|\int_0^T \int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\int_{\mathbb{T}^3}(L_3)g_{\delta,k}dtdxdvdk\right|$$
$$\lesssim \epsilon\|f\langle v\rangle^{N_1+2}\|_{L_t^\infty(H_x^2(L_v^2))}\|g_{\delta,k}\langle v\rangle^{\frac{\gamma}{2}}\|_{L_{t,x,k}^2(H_v^s)}^2$$
$$(5.88)\quad +C_\epsilon\|f\langle v\rangle^{N_1+2}\|_{L_t^\infty(H_x^2(L_v^2))}\|g_{\delta,k}\langle v\rangle^{N_3}\|_{L_{t,x,v,k}^2},$$

while in the case $s < \frac{1}{2}$ (taking $M_1 = |l + \frac{\gamma}{2}| + |\frac{\gamma}{2}| + \max\{|l-2|,|l-1|\}$, $M_2 = l + \frac{\gamma}{2}$ and $N_3 = \frac{\gamma}{2}$),

$$\left|\int_0^T \int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\int_{\mathbb{T}^3}(L_3)g_{\delta,k}dtdxdvdk\right|$$
$$\lesssim \epsilon\|f\langle v\rangle^{M_1+2}\|_{L_t^\infty(H_x^2(L_v^2))}\|g_{\delta,k}\langle v\rangle^{\frac{\gamma}{2}}\|_{L_{t,x,k}^2(H_v^s)}^2$$
$$(5.89)\quad +C_\epsilon\|f\langle v\rangle^{M_1+2}\|_{L_t^\infty(H_x^2(L_v^2))}\|g_{\delta,k}\langle v\rangle^{M_3}\|_{L_{t,x,v,k}^2}.$$

We now turn to consider the term containing (L_4). In the case $1 \leq |\alpha_1| + |\beta_1| \leq [\frac{N}{2}]$, one has $|\alpha_1| + |\beta_1| + 2 \leq N - 1$ and $|\alpha_2| + |\beta_2| \leq N - 1$, so that Theorem 2.1 gives (taking $N_1 = |\frac{\gamma}{2} + 2s| + |\frac{\gamma}{2}|$, $N_2 = \frac{\gamma}{2} + 2s$ and $N_3 = \frac{\gamma}{2}$)

$$\left|\int_0^T \int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\int_{\mathbb{T}^3}(L_4)g_{\delta,k}dtdxdvdk\right|$$
$$\lesssim \|(\partial_{\beta_1}^{\alpha_1} f)\langle v\rangle^{N_1}\|_{L_{t,x}^\infty(L_v^1)}\left\|\frac{\triangle_k(\partial_\beta^\alpha f)}{|k|^{\delta+\frac{3}{2}}}\langle v\rangle^{l+N_2}\right\|_{L_{t,x,k}^2(H_v^s)}\|g_{\delta,k}\langle v\rangle^{N_3}\|_{L_{t,x,k}^2(H_v^s)}$$
$$\lesssim \|(\partial_{\beta_1}^{\alpha_1} f)\langle v\rangle^{N_1+2}\|_{L_t^\infty(H_x^2(L_v^2))}\|(\partial_{\beta_2}^{\alpha_2} f)\langle v\rangle^{l+N_2}\|_{L_t^2(\dot{H}_x^\delta(H_v^s))}\|g_{\delta,k}\langle v\rangle^{N_3}\|_{L_{t,x,k}^2(H_v^s)}$$
$$(5.90)\lesssim \epsilon\|g_{\delta,k}\langle v\rangle^{\frac{\gamma}{2}}\|_{L_{t,x,k}^2(H_v^s)}^2 + C_\epsilon\|(\partial_{\beta_1}^{\alpha_1} f)\langle v\rangle^{N_1+2}\|_{L_t^\infty(H_x^2(L_v^2))}^2\|(\partial_{\beta_2}^{\alpha_2} f)\langle v\rangle^{l+N_2}\|_{L_t^2(\dot{H}_x^\delta(H_v^s))}^2.$$

In the case $|\alpha_1| + |\beta_1| \geq [\frac{N}{2}] + 1$, one has $|\alpha_2| + |\beta_2| + 2 + \delta + s \leq N$. Again by Theorem 2.1 (taking $N_1 = |\frac{\gamma}{2} + 2s| + |\frac{\gamma}{2}|$, $N_2 = \frac{\gamma}{2} + 2s$ and $N_3 = \frac{\gamma}{2}$),

$$\left|\int_0^T \int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\int_{\mathbb{T}^3}(L_4)g_{\delta,k}dtdxdvdk\right|$$
$$\lesssim \|(\partial_{\beta_1}^{\alpha_1} f)\langle v\rangle^{N_1}\|_{L_{t,x}^2(L_v^1)}\left\|\frac{\triangle_k(\partial_\beta^\alpha f)}{|k|^{\delta+\frac{3}{2}}}\langle v\rangle^{l+N_2}\right\|_{L_{t,x}^\infty(L_k^2(H_v^s))}\|g_{\delta,k}\langle v\rangle^{N_3}\|_{L_{t,x,k}^2(H_v^s)}$$
$$\lesssim \|(\partial_{\beta_1}^{\alpha_1} f)\langle v\rangle^{N_1+2}\|_{L_t^2(H_x^2(L_v^2))}\|(\partial_{\beta_2}^{\alpha_2} f)\langle v\rangle^{l+N_2}\|_{L_t^\infty(H_x^{2+\delta}(H_v^s))}\|g_{\delta,k}\langle v\rangle^{N_3}\|_{L_{t,x,k}^2(H_v^s)}$$
$$(5.91)\lesssim \epsilon\|g_{\delta,k}\langle v\rangle^{\frac{\gamma}{2}}\|_{L_{t,x,k}^2(H_v^s)}^2 + C_\epsilon\|(\partial_{\beta_1}^{\alpha_1} f)\langle v\rangle^{N_1+2}\|_{L_t^2(H_x^2(L_v^2))}^2\|(\partial_{\beta_2}^{\alpha_2} f)\langle v\rangle^{l+N_2}\|_{L_t^\infty(H_x^{2+\delta}(H_v^s))}^2.$$

We point out that the estimates for $\int_0^T \int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\int_{\mathbb{T}^3}(L_i)g_{\delta,k}dtdxdvdk$ $(i = 5, 6, 7)$ are similar to those for $\int_0^T \int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\int_{\mathbb{T}^3}(L_4)g_{\delta,k}dtdxdvdk$, so we omit them.

Therefore, if we notice $h(0) \in L^2_v(\dot{H}^\delta_x)$ and $h \in L^2_{t,v}(\dot{H}^\delta_x)$ imply $g_{\delta,k}(0) \in L^2_{x,v,k}$ and $g_{\delta,k} \in L^2_{t,x,v,k}$, we finally get $g_{\delta,k} \in L^2_{t,x,k}(H^s_v)$ by choosing ϵ appeared in the above estimates sufficiently small. This completes the proof of Lemma 5.4.

∎

The following lemma shows that h can gain another $\frac{s}{4(4+s)}$ derivative in x variable based on Lemma 5.3.

Lemma 5.5. *Let $N \geq 5$ be a given integer, and let f be a smooth solution to the Boltzmann equation (1.1). We suppose that for any $T > 0$ and any $l \geq 0$, $h \in L^\infty_t(L^2_{x,v}) \cap L^2_{t,v}(\dot{H}^\delta_x)$ and $h(0) \in L^2_v(\dot{H}^\delta_x)$ where h is defined in Proposition 5.1, $\delta = \frac{s}{4(4+s)}$. Then we have $h \in L^2_{t,v}(\dot{H}^{2\delta}_x)$.*

Proof: Thanks to the fact

$$\int_{\mathbb{T}^3} |\widehat{g_{\delta,k}}(m)| dk = C|m|^{2\delta}|\hat{h}(m)|^2,$$

we know that in order to get $h \in L^2_{t,v}(\dot{H}^{2\delta}_x)$, we can equivalently prove $g_{\delta,k} \in L^2_{t,v,k}(\dot{H}^\delta_x)$. We still only consider the case $|\beta| \geq 1$. The equation for $g_{\delta,k}$ can be rewritten as:

$$\partial_t g_{\delta,k} + v \cdot \nabla_x g_{\delta,k}$$
$$= -\beta_i \left[\frac{\triangle_k(\partial^{\alpha+e_i}_{\beta-e_i} f)}{|k|^{\delta+\frac{3}{2}}} \right] \langle v \rangle^l + \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} C^\alpha_{\alpha_1} C^\beta_{\beta_1} Q\left(\partial^{\alpha_1}_{\beta_1} f(x+k), \frac{\triangle_k((\partial^{\alpha_2}_{\beta_2} f)(x))\langle v \rangle^l}{|k|^{\delta+\frac{3}{2}}} \right)$$
$$+ \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} C^\alpha_{\alpha_1} C^\beta_{\beta_1} \left[Q\left(\partial^{\alpha_1}_{\beta_1} f(x+k), \frac{\triangle_k(\partial^{\alpha_2}_{\beta_2} f(x))}{|k|^{\delta+\frac{3}{2}}} \right) \langle v \rangle^l \right.$$
$$\left. -Q\left(\partial^{\alpha_1}_{\beta_1} f(x+k), \frac{\triangle_k((\partial^{\alpha_2}_{\beta_2} f)(x))\langle v \rangle^l}{|k|^{\delta+\frac{3}{2}}} \right) \right]$$
$$+ \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} C^\alpha_{\alpha_1} C^\beta_{\beta_1} Q\left(\frac{\triangle_k(\partial^{\alpha_1}_{\beta_1} f)}{|k|^{\delta+\frac{3}{2}}}, (\partial^{\alpha_2}_{\beta_2} f)\langle v \rangle^l \right)$$
$$+ \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} C^\alpha_{\alpha_1} C^\beta_{\beta_1} \left[Q\left(\frac{\triangle_k(\partial^{\alpha_1}_{\beta_1} f)}{|k|^{\delta+\frac{3}{2}}}, \partial^{\alpha_2}_{\beta_2} f \right) \langle v \rangle^l - Q\left(\frac{\triangle_k(\partial^{\alpha_1}_{\beta_1} f)}{|k|^{\delta+\frac{3}{2}}}, (\partial^{\alpha_2}_{\beta_2} f)\langle v \rangle^l \right) \right]$$

$$(5.92) \stackrel{\text{def}}{=} (M_1) + (M_2) + (M_3) + (M_4) + (M_5).$$

It is easy to see that $(M_1) \in L^2_{t,x,v,k}$. Theorem 2.1 gives (taking $N_1 = N_2 = \gamma + 2s$, $N_3 = 0$)

$$\left\| Q\left(\partial^{\alpha_1}_{\beta_1} f(x+k), \frac{\triangle_k((\partial^{\alpha_2}_{\beta_2} f)(x))\langle v \rangle^l}{|k|^{\delta+\frac{3}{2}}} \right) \right\|_{H^{-s}_v}$$
$$\lesssim \|(\partial^{\alpha_1}_{\beta_1} f)(x+k)\langle v \rangle^{\gamma+2s+2}\|_{L^2_v} \left\| \frac{\triangle_k((\partial^{\alpha_2}_{\beta_2} f)(x))}{|k|^{\delta+\frac{3}{2}}} \langle v \rangle^{\gamma+2s+l} \right\|_{H^s_v}.$$

Then in the case $|\alpha_1| + |\beta_1| \leq \left[\frac{N}{2}\right]$, on has $|\alpha_1| + |\beta_1| + 2 \leq N - 1$, so that

$$\left\| Q\left(\partial^{\alpha_1}_{\beta_1} f(x+k), \frac{\triangle_k((\partial^{\alpha_2}_{\beta_2} f)(x))\langle v \rangle^l}{|k|^{\delta+\frac{3}{2}}} \right) \right\|_{L^2_{t,x,k}(H^{-s}_v)}$$
$$\lesssim \|(\partial^{\alpha_1}_{\beta_1} f)\langle v \rangle^{\gamma+2s+2}\|_{L^\infty_t(H^2_x(L^2_v))} \|(\partial^{\alpha_2}_{\beta_2} f)\langle v \rangle^{\gamma+2s+l}\|_{L^2_t(\dot{H}^\delta_x(H^s_v))}.$$

while in the case $|\alpha_1| + |\beta_1| \geq \left[\frac{N}{2}\right] + 1$, one has $|\alpha_2| + |\beta_2| + 2 + \delta + s \leq N$, so that

$$\left\| Q\left(\partial_{\beta_1}^{\alpha_1} f(x+k), \frac{\triangle_k((\partial_{\beta_2}^{\alpha_2} f)(x))\langle v\rangle^l)}{|k|^{\delta+\frac{3}{2}}} \right) \right\|_{L^2_{t,x,k}(H_v^{-s})}$$
$$\lesssim \quad \|(\partial_{\beta_1}^{\alpha_1} f)\langle v\rangle^{\gamma+2s+2}\|_{L^2_{t,x,v}} \|(\partial_{\beta_2}^{\alpha_2} f)\langle v\rangle^{\gamma+2s+l}\|_{L^\infty_t(H_x^{2+\delta}(H_v^s))}.$$

We thus obtain that $(M_2) \in L^2_{t,x,k}(H_v^{-s})$ by Lemma 5.4. Analogously, we can get (M_3), (M_4), $(M_5) \in L^2_{t,x,k}(H_v^{-s})$. We now know that the right-hand side of equation (5.92) belongs to $L^2_{t,x,k}(H_v^{-s})$, and then we conclude $g_{\delta,k} \in L^2_{t,v,k}(\dot{H}_x^\delta)$ thanks to Lemma 5.1. $\blacksquare$

Applying Lemmas 5.2 and 5.3, we get $h \in L^2([0,T]; L^2_v(\dot{H}_x^\delta))$ with $\delta = \frac{s}{4(4+s)}$. Then for any $t_* \in (0,T)$, we can find some time $t_1 \in (0,t_*)$ such that $h(t_1) \in L^2_v(\dot{H}_x^\delta)$. So we can use Lemmas 5.4 and 5.5 to obtain $h \in L^2([t_1,T]; L^2_v(\dot{H}_x^{2\delta}))$. As a consequence, we can find some time $t_2 \in (t_1, t_*)$ such that $h(t_2) \in L^2_v(\dot{H}_x^{2\delta})$. If we repeat this procedure (by using Lemmas 5.4 and 5.5) $m-1$ times such that $m\delta \geq 1$, we obtain $h \in L^2([t_{m-1},T]; L^2_v(H_x^1))$ for some time $t_{m-1} \in (t_{m-2}, t_*)$, and we can find some time $t_m \in (t_{m-1}, t_*)$ such that $h(t_m) \in L^2_v(H_x^1)$. Therefore, thanks to Lemma 5.2, we finally get $h \in L^\infty([t_*, T]; L^2_v(H_x^1))$.

5.3. Gain regularity in v variable.

In this subsection we shall improve regularity in v variable. We begin with

Lemma 5.6. *Let $N \geq 5$ be a given integer, and let f be a smooth solution to the Boltzmann equation* (1.1)*. We suppose that for any $T > 0$ and any $l \geq 0$, $h \in L^\infty_t(H_x^1(L_v^2)) \cap L^2_{t,x}(H_v^s)$ and $h(0) \in L^2_x(H_v^s)$ where h is defined in Proposition 5.1. Then we have $h \in L^\infty_t(L^2_x(H_v^s)) \cap L^2_{t,x}(H_v^{2s})$.*

Proof: Let τ_u be the translation operator in the v variable by u, then one has

$$\tau_u f(t,x,v) = f(t,x,v+u) - f(t,x,v).$$

We denote the finite difference of f in the v variable by

$$(5.93) \qquad\qquad \triangle_u f(t,x,v) = \tau_u f(t,x,v) - f(t,x,v).$$

If we define $g_{s,u}(t,x,v) = \triangle_u h(t,x,v)|u|^{-s-\frac{3}{2}}$, then we know that in order to get $h \in L^\infty_t(L^2_x(H_v^s)) \cap L^2_{t,x}(H_v^{2s})$, we can equivalently prove $g_{s,u} \in L^\infty_t(L^2_{x,v,u}) \cap L^2_{t,x,u}(H_v^s)$. Since $h \in L^\infty_t(H_x^1(L_v^2)) \cap L^2_{t,x}(H_v^s)$, we can restrict the integral domain of variable u to $\mathbb{B}_1 = \{u \in \mathbb{R}^3 : |u| \leq 1\}$.

We still only consider the case $|\beta| \geq 1$. Firstly, for any function $p(v)$ and $q(v)$, one has

$$\triangle_u(p(v)q(v)) = p(v+u)\triangle_u q(v) + p(v)\triangle_u q(v).$$

Secondly, the translation invariance of the collision operator with respect to the variable v gives that (see [28] for instance)

$$\tau_u Q(f,g) = Q(\tau_u f, \tau_u g).$$

Applying these two equalities, we get the equation for $g_{s,u}$ as follows:

$$\partial_t g_{s,u} + v \cdot \nabla_x g_{s,u}$$

$$= -u \cdot \nabla_x h(v+u)|u|^{-s-\frac{3}{2}} - \beta_i \left[(\partial_{\beta-e_i}^{\alpha+e_i} f)(v+u) \right] \frac{\triangle_u \langle v \rangle^l}{|u|^{s+\frac{3}{2}}}$$

$$-\beta_i \left[\frac{\triangle_u (\partial_{\beta-e_i}^{\alpha+e_i} f)}{|u|^{\delta+\frac{3}{2}}} \right] \langle v \rangle^l + \left[\partial_\beta^\alpha Q(f(v+u), f(v+u)) \right] \frac{\triangle_u \langle v \rangle^l}{|u|^{s+\frac{3}{2}}}$$

$$+ \left[\partial_\beta^\alpha Q \left(f(v+u), \frac{\triangle_u f}{|u|^{s+\frac{3}{2}}} \right) \right] \langle v \rangle^l + \left[\partial_\beta^\alpha Q \left(\frac{\triangle_u f}{|u|^{s+\frac{3}{2}}}, f \right) \right] \langle v \rangle^l$$

$$\stackrel{\text{def}}{=} \quad (P_1) + (P_2) + (P_3) + (P_4) + (P_5) + (P_6). \tag{5.94}$$

Multiplying the above equation by $g_{s,u}$, then integrating on (t, x, v, u) in the domain $[0, T] \times \mathbb{T}^3 \times \mathbb{R}^3 \times \mathbb{B}_1$. Similar to the estimates (5.72) and (5.73), we yield

$$\int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{B}_1} (\partial_t g_{s,u}) g_{s,u} dt dx dv du = \frac{1}{2} \left(\|g_{s,u}(T)\|_{L^2_{x,v,u}}^2 - \|g_{s,u}(0)\|_{L^2_{x,v,u}}^2 \right) \tag{5.95}$$

and
(5.96)
$$\int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{B}_1} (v \cdot \nabla_x g_{s,u}) g_{s,u} dt dx dv du = \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{B}_1} v \cdot \left(\int_{\mathbb{T}^3} \nabla_x (g_{s,u}^2) dx \right) dt dv du = 0.$$

Cauchy-Schwartz inequality gives

$$\left| \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{B}_1} (P_1) g_{s,u} dt dx dv du \right| \lesssim \int_0^T \int_{\mathbb{T}^3} \left(\int_{\mathbb{B}_1} |u|^{-s-\frac{1}{2}} \|h\|_{\dot{H}_x^1} \|g_{s,u}\|_{L^2_v} du \right) dt dx$$

$$\lesssim \|h\|_{L^2_{t,v}(\dot{H}_x^1)} \|g_{s,u}\|_{L^2_{t,x,u,v}}. \tag{5.97}$$

We have that

$$|\triangle_u \langle v \rangle^l| = |\langle v + u \rangle^l - \langle v \rangle^l| \lesssim \int_0^1 \langle v + \theta u \rangle^{l-1} d\theta |u| \lesssim \langle v + u \rangle^{l-1} |u|, \tag{5.98}$$

where the last inequality holds due to the fact $|u| \leq 1$. Then,

$$\left| \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{B}_1} (P_2) g_{s,u} dt dx dv du \right|$$

$$\lesssim \int_0^T \int_{\mathbb{T}^3} \left(\int_{\mathbb{B}_1} |u|^{-s-\frac{1}{2}} \|(\partial_{\beta-e_i}^{\alpha+e_i} f)(v+u) \langle v + u \rangle^{l-1}\|_{L^2_v} \|g_{s,u}\|_{L^2_v} du \right) dt dx$$

$$\lesssim \|(\partial_{\beta-e_i}^{\alpha+e_i} f) \langle v \rangle^{l-1}\|_{L^2_{t,x,v}} \|g_{s,u}\|_{L^2_{t,x,u,v}}. \tag{5.99}$$

Again by Cauchy-Schwartz inequality, we get

$$\left| \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{B}_1} (P_3) g_{s,u} dt dx dv du \right| \lesssim \|\partial_{\beta-e_i}^{\alpha+e_i} f\|_{L^2_{t,x}(H^s_v)} \|g_{s,u} \langle v \rangle^l\|_{L^2_{t,x,u,v}}. \tag{5.100}$$

We now turn to treat the term $\int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{B}_1} (P_5) g_{s,u} dt dx dv du$. We write

$$(P_5) = Q(f(v+u), g_{s,u}) - Q\left(f(v+u), (\partial_\beta^\alpha f)(v+u)\frac{\triangle_u \langle v \rangle^l}{|u|^{s+\frac{3}{2}}}\right)$$

$$+ \left[Q\left(f(v+u), \frac{\triangle_u(\partial_\beta^\alpha f)}{|u|^{s+\frac{3}{2}}}\right)\langle v \rangle^l - Q\left(f(v+u), \frac{\triangle_u(\partial_\beta^\alpha f)\langle v \rangle^l}{|u|^{s+\frac{3}{2}}}\right)\right]$$

$$+ \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta \\ |\alpha_1|+|\beta_1|\geq 1}} C_{\alpha_1}^\alpha C_{\beta_1}^\beta Q\left((\partial_{\beta_1}^{\alpha_1} f)(v+u), \frac{\triangle_u((\partial_{\beta_2}^{\alpha_2} f)\langle v \rangle^l)}{|u|^{s+\frac{3}{2}}}\right)$$

$$- \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta \\ |\alpha_1|+|\beta_1|\geq 1}} C_{\alpha_1}^\alpha C_{\beta_1}^\beta Q\left((\partial_{\beta_1}^{\alpha_1} f)(v+u), (\partial_{\beta_2}^{\alpha_2} f)(v+u)\frac{\triangle_u \langle v \rangle^l}{|u|^{s+\frac{3}{2}}}\right)$$

$$+ \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta \\ |\alpha_1|+|\beta_1|\geq 1}} C_{\alpha_1}^\alpha C_{\beta_1}^\beta \left[Q\left((\partial_{\beta_1}^{\alpha_1} f)(v+u), \frac{\triangle_u(\partial_{\beta_2}^{\alpha_2} f)}{|u|^{s+\frac{3}{2}}}\right)\langle v \rangle^l\right.$$

$$\left. - Q\left((\partial_{\beta_1}^{\alpha_1} f)(v+u), \frac{\triangle_u(\partial_{\beta_2}^{\alpha_2} f)\langle v \rangle^l}{|u|^{s+\frac{3}{2}}}\right)\right]$$

$$(5.101) \qquad \overset{\text{def}}{=} (P_5)_1 + (P_5)_2 + (P_5)_3 + (P_5)_4 + (P_5)_5 + (P_5)_6.$$

Applying Theorem 3.1, we get

$$\left|\int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{B}_1} (P_5)_1 g_{s,u} dt dx dv du\right|$$

$$\lesssim -(C_f - \epsilon\|f\langle v \rangle^{|\gamma|+2}\|_{L_t^\infty(H_x^2(L_v^2))})\|g_{s,u}\langle v \rangle^{\frac{\gamma}{2}}\|_{L_{t,x,u}^2(H_v^s)}^2 + (C_f^{1-2s}\|f\langle v \rangle^{\tilde{\gamma}+2}\|_{L_t^\infty(H_x^2(L_v^2))}^{2s}$$

$$(5.102) \quad + \|f\langle v \rangle^{\tilde{\gamma}+2}\|_{L_t^\infty(H_x^2(L_v^2))} + C_\epsilon\|f\langle v \rangle^{|\gamma|+2}\|_{L_t^\infty(H_x^2(L_v^2))})\|g_{s,u}\langle v \rangle^{\frac{\gamma}{2}}\|_{L_{t,x,v,u}^2}^2.$$

Remembering $|u| \leq 1$, We then get by Theorem 2.1 and (5.98) that

$$\left|\int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{B}_1} (P_5)_2 g_{s,u} dt dx dv du\right|$$

$$(5.103) \quad \epsilon\|g_{s,u}\langle v \rangle^{\frac{\gamma}{2}}\|_{L_{t,x,u}^2(H_v^s)}^2 + C_\epsilon\|f\langle v \rangle^{|\frac{\gamma}{2}+2s|+|\frac{\gamma}{2}|+2}\|_{L_t^\infty(H_x^2(L_v^2))}^2 \|(\partial_\beta^\alpha f)\langle v \rangle^{l-1+\frac{\gamma}{2}+2s}\|_{L_{t,x}^2(H_v^s)}^2.$$

As for the terms $\int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{B}_1} (P_5)_i g_{s,u} dt dx dv du$ $(i = 3, 4)$, we claim that we can bound them by the quantities which have been controlled and $\epsilon\|g_{s,u}\langle v \rangle^{\frac{\gamma}{2}}\|_{L_{t,x,u}^2(H_v^s)}^2$. The proof is very close to the estimates for the terms $\int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{T}^3} (L_i) g_{\delta,k} dt dx dv dk$ $(i = 3, 4)$ in Lemma 5.4, so we omit it. And we can analogously bound the terms $\int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{B}_1} (P_5)_j g_{s,u} dt dx dv du$ for $j = 5, 6$. The estimates for $\int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \int_{\mathbb{B}_1} (P_k) g_{s,u} dt dx dv du$ $(k = 4, 6)$ are similar.

Therefore, the lemma is proved by taking ϵ appeared small enough. ∎

We now finish the proof of Proposition 5.1. According to the result of Subsection 5.2, one has for any $t_* \in (0, \tau)$, $h \in L^\infty([t_*, T]; L_v^2(H_x^1)) \cap L^2([t_*, T]; L_v^2(H_x^s))$. Then we can find some time $\tau_1 \in (t_*, \tau)$ such that $h(\tau_1) \in L_x^2(H_v^s)$. Thanks to Lemma 5.6, we get $h \in L^\infty([\tau_1, T]; L_x^2(H_v^s)) \cap L^2([\tau_1, T]; L_x^2(H_v^{2s}))$. As a consequence, we can find some time $\tau_2 \in (\tau_1, \tau)$ such that $h(\tau_2) \in L_x^2(H_v^{2s}))$. Repeating this procedure (by using Lemma 5.6) m times such that $ms \geq 1$, we finally obtain $h \in L^\infty([\tau_m, T]; L_x^2(H_v^1)) \cap L^2([\tau_m, T]; L_x^2(H_v^{(m+1)s}))$ (for

some time $\tau_m \in (\tau_{m-1}, \tau))$, which implies $h \in L^\infty([\tau, T]; L_x^2(H_v^1))$. Since we already have $h \in L^\infty([\tau, T]; L_v^2(H_x^1))$, Proposition 5.1 is therefore proven. $\blacksquare$

5.4. **Proof of Theorem 1.2.** We now end up the proof of Theorem 1.2. By applying Proposition 5.1 repeatedly, we get that for any $l \geq 0$ and any $0 < \tau < T < +\infty$,

$$(5.104) \qquad f\langle v\rangle^l \in L^\infty([\tau, T]; H_{x,v}^\infty).$$

We claim that for any nonnegative integer n,

$$(5.105) \qquad (\partial_t^n f)\langle v\rangle^l \in L^\infty([\tau, T]; H_{x,v}^\infty).$$

We shall prove this by induction on n. Thanks to (5.104), this is true for $n = 0$. If we assume that (5.105) holds for any integer $k \leq n$, then for any $l \geq 0$ and any multi-indices α and β,

$$
\begin{aligned}
(\partial_\beta^\alpha \partial_t^{n+1} f)\langle v\rangle^l &= -\partial_\beta^\alpha[v \cdot \nabla_x(\partial_t^n f)]\langle v\rangle^l \\
&\quad + \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} \sum_{m=0}^n C_{\alpha_1}^\alpha C_{\beta_1}^\beta C_m^n Q(\partial_{\beta_1}^{\alpha_1}\partial_t^m f, \partial_{\beta_2}^{\alpha_2}\partial_t^{n-m} f)\langle v\rangle^l \\
&\stackrel{\text{def}}{=} (R_1) + (R_2).
\end{aligned}
$$

It is obvious from the induction hypothesis that $(R_1) \in L^\infty([\tau, T]; L_{x,v}^2)$. For any nonnegative function $W \in L^1([\tau, T]; L_{x,v}^2)$, we shall consider the quantity

$$\langle Q(U, V)\langle v\rangle^l, W\rangle_v,$$

where we set $U = \partial_{\beta_1}^{\alpha_1}\partial_t^m f$ and $V = \partial_{\beta_2}^{\alpha_2}\partial_t^{n-m} f$ for simplicity. We write

$$
\begin{aligned}
&\langle Q(U, V)\langle v\rangle^l, W\rangle_v \\
&= \int_{\mathbb{R}^6} dv dv_* \int_{\mathbb{S}^2} B(v - v_*, \sigma) U_* V (W'\langle v'\rangle^l - W\langle v\rangle^l) d\sigma \\
&= \int_{\mathbb{R}^6} dv dv_* \int_{\mathbb{S}^2} B(v - v_*, \sigma) U_* (V'W'\langle v'\rangle^l - VW\langle v\rangle^l) d\sigma \\
&\quad + \int_{\mathbb{R}^6} dv dv_* \int_{\mathbb{S}^2} B(v - v_*, \sigma) U_* W'(V\langle v\rangle^l - V'\langle v'\rangle^l) d\sigma \\
&\quad + \int_{\mathbb{R}^6} dv dv_* \int_{\mathbb{S}^2} B(v - v_*, \sigma) U_* V'W'(\langle v'\rangle^l - \langle v\rangle^l) d\sigma \\
&\quad + \int_{\mathbb{R}^6} dv dv_* \int_{\mathbb{S}^2} B(v - v_*, \sigma) U_* W'(V - V')(\langle v'\rangle^l - \langle v\rangle^l) d\sigma \\
(5.106) \qquad &\stackrel{\text{def}}{=} (S_1) + (S_2) + (S_3) + (S_4).
\end{aligned}
$$

We only give the estimates for (S_2), and those for $(S_1), (S_3)$ and (S_4) are analogous. Applying Taylor expansion formula up to order 2, we get

$$V\langle v\rangle^l - V'\langle v'\rangle^l = (v - v') \cdot \nabla_v(V'\langle v'\rangle^l) + \int_0^1 (v - v') \otimes (v - v') : \nabla_v^2(V(\gamma(\kappa))\langle\gamma(\kappa)\rangle^l) d\kappa,$$

where $\gamma(\kappa) = \kappa v' + (1 - \kappa)v$. If we change the variables from v to v', and then use (2.36), we deduce that

$$(5.107) \qquad \int_{\mathbb{R}^6} dv dv_* \int_{\mathbb{S}^2} B(v - v_*, \sigma) U_* W'(v - v') \cdot \nabla_v(V'\langle v'\rangle^l) d\sigma = 0.$$

So we only need to study the term

$$\int_0^1 d\kappa \int_{\mathbb{R}^6} dv dv_* \int_{\mathbb{S}^2} B(v - v_*, \sigma) U_* W'(v - v') \otimes (v - v') : \nabla_v^2(V(\gamma(\kappa))\langle\gamma(\kappa)\rangle^l) d\sigma \stackrel{\text{def}}{=} (Y).$$

Thanks to the fact $|v - v'|^2 = |v - v_*|^2 \sin^2 \frac{\theta}{2}$, we get by Cauchy-Schwartz inequality that

$$|(Y)|$$
$$\lesssim \left(\int_{\mathbb{R}^6} dv dv_* \int_{\mathbb{S}^2} B(v - v_*, \sigma) |v - v_*|^{-\gamma} U_* W'^2 \sin^2 \frac{\theta}{2} d\sigma \right)^{\frac{1}{2}}$$
$$\times \left(\int_0^1 d\kappa \int_{\mathbb{R}^6} dv dv_* \int_{\mathbb{S}^2} B(v - v_*, \sigma) |v - v_*|^{4+\gamma} U_* |\nabla_v^2 (V(\gamma(\kappa)) \langle \gamma(\kappa) \rangle^l)|^2 \sin^2 \frac{\theta}{2} d\sigma \right)^{\frac{1}{2}}$$
$$\overset{\text{def}}{=} (Y_1)^{\frac{1}{2}} (Y_2)^{\frac{1}{2}}.$$

Changing the variables from v to v', we get from (2.26) with $\kappa = 1$ that

$$(Y_1) \lesssim \int_0^{\frac{\pi}{2}} \theta^{1-2s} d\theta \int_{\mathbb{R}^6} U_* W'^2 dv' dv_* \lesssim \|U \langle v \rangle^2\|_{L_v^2} \|W\|_{L_v^2}^2.$$

Similarly, changing the variables from v to $\gamma(\kappa) = u$, we get from (2.26) that

$$(Y_2) \lesssim \int_0^{\frac{\pi}{2}} \theta^{1-2s} d\theta \int_{\mathbb{R}^6} U_* |u - v_*|^{4+2\gamma} |\nabla_v^2 (V(u) \langle u \rangle^l)|^2 du dv_*.$$

Noticing $\gamma + 2s > -1$ implies $4 + 2\gamma > -2$, we get in the case $4 + 2\gamma \geq 0$,

$$(Y_2) \lesssim \|U \langle v \rangle^{6+2\gamma}\|_{L_v^2} \|V \langle v \rangle^{l+4+2\gamma}\|_{H_v^2}^2,$$

while in the case $-2 < 4 + 2\gamma < 0$,

$$(Y_2) \lesssim \int_{\mathbb{R}^6} U_* (1 + |u - v_*|^{4+2\gamma} \mathbf{1}_{|v-v_*| \leq 1}) |\nabla_v^2 (V(u) \langle u \rangle^l)|^2 du dv_*$$
$$\lesssim (\|U\|_{L_v^1} + \|U\|_{L_v^\infty}) \|V \langle v \rangle^l\|_{H_v^2} \lesssim \|U \langle v \rangle^2\|_{H_v^2} \|V \langle v \rangle^l\|_{H_v^2}.$$

Then we arrive at

(5.108)
$$|(Y)| \lesssim (\|U \langle v \rangle^{6+2\gamma}\|_{L_v^2} + \|U \langle v \rangle^2\|_{H_v^2}) \|V \langle v \rangle^{l+6+2\gamma}\|_{H_v^2} \|W\|_{L_v^2},$$

which together with the induction hypothesis implies $(S_2) \in L^\infty([\tau, T]; L_{x,v}^2)$. We now conclude that (5.105) holds true, that is, for any $l \geq 0$ and any $0 < \tau < T < +\infty$,

$$f \langle v \rangle^l \in W^{\infty,\infty}([\tau, T]; H_{x,v}^\infty).$$

We point out Theorem 1 of [37] shows that for any $l \geq 0$ and any integer $N \geq 5$, $f \langle v \rangle^l \in L^\infty([0, +\infty); H_{x,v}^N)$ as long as $\|F_0 \langle v \rangle^l\|_{H_{x,v}^N} \leq \eta_0$ for some $\eta_0 > 0$. Then it is easy to check that our estimates established up to now can be made independent on the time T, so that we actually obtain

$$f \langle v \rangle^l \in W^{\infty,\infty}([\tau, \infty); H_{x,v}^\infty).$$

This completes the proof of Theorem 1.2. ∎

Remark 5.2. *We note that for the purpose of having a completely rigorous proof of Theorem 1.2, all the estimates in the proof should actually be made on a version of the Boltzmann equation (1.1) with smooth data and then extended to the solution under consideration by a passage to the limit. This creates no difficulty.*

6. Appendix

In this appendix, we give the proof to the following proposition:

Proposition 6.1. *Suppose $\gamma > 0$ and $0 < s < 1$. Then for any $R \geq 1$ and any smooth function χ_R defined as $\chi_R = \chi(\frac{\cdot}{R})$ with $0 \leq \chi \leq 1, \chi = 1$ on B_1 and $\mathrm{supp}(\chi) \subset B_2$, there holds*

$$\|f\|_{\dot{H}^s(\mathbb{R}_v^3)} \leq \|f \chi_R\|_{\dot{H}^s(\mathbb{R}_v^3)} + R^{-\frac{\gamma}{2}} \|f \langle v \rangle^{\frac{\gamma}{2}}\|_{\dot{H}^s(\mathbb{R}_v^3)} + \|f \langle v \rangle^{\frac{\gamma}{2}}\|_{L^2(\mathbb{R}_v^3)}$$

Proof: We first recall that for $0 < s < 1$ and smooth function ϕ, there hold

$$\|f\phi\|_{\dot{H}^s}^2 = \int_{\mathbb{R}^6} \frac{|f\phi(v) - f\phi(w)|^2}{|v-w|^{3+2s}} dvdw,$$

and

$$\|f\|_{\dot{H}^s(\mathbb{R}_v^3)} \leq \|f\chi_R\|_{\dot{H}^s(\mathbb{R}_v^3)} + \|f(1-\chi_R)\|_{\dot{H}^s(\mathbb{R}_v^3)} + \|f\|_{L^2(\mathbb{R}_v^3)}.$$

We also note that

$$|f\phi(v) - f\phi(w)| = |\phi(v)(f(v) - f(w)) + f(w)(\phi(v) - \phi(w))|.$$

Then on the one hand, one may deduce that

$$\|f\phi\|_{\dot{H}^s}^2 \;\lesssim\; \|f\phi\|_{L^2}^2 + \int_{\mathbb{R}^6} \frac{|f(v) - f(w)|^2}{|v-w|^{3+2s}}|\phi(v)|^2 \mathbf{1}_{|v-w|\leq 1} dvdw$$
$$+ \|\nabla\phi\|_{L^\infty}^2 \|f\|_{L^2}^2,$$

which implies that

$$\|f(1-\chi_R)\|_{\dot{H}^s}^2 \;\lesssim\; \|f(1-\chi_R)\|_{L^2}^2 + \int_{\mathbb{R}^6} \frac{|f(v) - f(w)|^2}{|v-w|^{3+2s}}|(1-\chi_R)|^2 \mathbf{1}_{|v-w|\leq 1} dvdw$$

$$(6.109) \qquad\qquad + \frac{1}{R}\|\nabla\chi\|_{L^\infty}^2 \|f\|_{L^2}^2.$$

On the other hand, one has

$$\|f\phi\|_{\dot{H}^s}^2 \;\gtrsim\; \frac{1}{2}\int_{\mathbb{R}^6} \frac{|f(v) - f(w)|^2}{|v-w|^{3+2s}}|\phi(v)|^2 \mathbf{1}_{|v-w|\leq 1} dvdw$$
$$- \int_{\mathbb{R}^6} \frac{|\phi(v) - \phi(w)|^2}{|v-w|^{3+2s}}|f(w)|^2 \mathbf{1}_{|v-w|\leq 1} dvdw.$$

Observing that

$$|\langle v\rangle^{\frac{\gamma}{2}} - \langle w\rangle^{\frac{\gamma}{2}}|^2 \mathbf{1}_{|v-w|\leq 1} \lesssim |v-w|^2 \langle w\rangle^{\gamma-2},$$

one may obtain that

$$(6.110) \quad \|f\langle v\rangle^{\frac{\gamma}{2}}\|_{\dot{H}^s}^2 \gtrsim \frac{1}{2}\int_{\mathbb{R}^6} \frac{|f(v) - f(w)|^2}{|v-w|^{3+2s}}\langle v\rangle^\gamma \mathbf{1}_{|v-w|\leq 1} dvdw - \|f\langle v\rangle^{\frac{\gamma}{2}-1}\|_{L^2}^2.$$

Thanks to the fact

$$|1 - \chi_R|^2 \leq \frac{|v|^\gamma}{R^\gamma},$$

(6.109) and (6.110) will lead to the proposition.   ∎

References

[1] R. Alexandre, Integral estimates for a linear singular operator linked with the Boltzmann operator. I. Small singularities $0 < \mu < 1$, *Indiana Univ. Math. J.* 55 (2006), no. 6, 1975-2021.

[2] R. Alexandre, A Review of Boltzmann Equation with Singular Kernels, *Kinet. Relat. Models* 2 (2009), no. 4, 551-646.

[3] R. Alexandre, Sur le taux de dissipation d'entropie sans troncature angulaire. *C. R. Acad. Sci. Paris Serie. Math.* 326, 3 (1998), 311-315.

[4] R. Alexandre, From Boltzmann to Landau, preprint, 1989.

[5] R. Alexandre, L. Desvillettes, C. Villani, and B. Wennberg, Entropy dissipation and long- range interactions, *Arch. Ration. Mech. Anal.* 152 (2000), no. 4, 327-355.

[6] R. Alexandre and L. He, Integral estimates for a linear singular operator linked with the Boltzmann operator. II. High singularities $1 \leq \mu < 2$ (2008), no.4, 491–513.

[7] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang, Uncertainty principle and kinetic equations, *J. Funct. Anal.* 255 (2008), no. 8, 2013-2066.

[8] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang, Regularizing effect and local existence for non-cutoff Boltzmann equation, to appear in *Arch. Ration. Mech. Anal.*

[9] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang, Global existence and full regularity of the Boltzmann equation without angular cutoff, preprint 2010.

[10] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang, The Boltzmann equation without angular cutoff in the whole space I: an essential coercivity estimate, preprint 2010.

[11] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang, The Boltzmann equation without angular cutoff in the whole space I: Soft potential, preprint 2010.

[12] R. Alexandre and M. El Safadi, Littlewood-Paley theory and regularity issues in Boltzmann homogeneous equations. I. Non-cutoff case and Maxwellian molecules, *Math. Models Methods Appl. Sci.* 15 (2005), no. 6, 907-920.

[13] R. Alexandre and C. Villani, On the Boltzmann equation for long-range interactions, *Comm. Pure Appl. Math.*, **55**, 1 (2002), 30-70.

[14] R. Alexandre and C. Villani, On the Landau approximation in plasma physics, *Ann. Inst. H. Poincaré, Anal. Non Linéaire*, **21**, 1 (2004), 61-95.

[15] A. V. Bobylev, The theory of the nonlinear spatially uniform Boltzmann equation for Maxwell molecules, Mathematical physics reviews, Vol. 7, Soviet Sci. Rev. Sect. C Math. Phys. Rev., vol. 7, Harwood Academic Publ., Chur, 1988, pp. 111-233.

[16] F. Bouchut, Hypoelliptic regularity in kinetic equations, *J. Math. Pure Appl.* 81 (2002) 1135-1159.

[17] F. Bouchut and L. Desvillettes, A proof of the smoothing properties of the positive part of Boltzmann's kernel, *Rev. Mat. Iberoamericana*, **14**, 1 (1998), 47-61.

[18] F. Bouchut and L. Desvillettes, Averaging lemmas without time Fourier transform and application to discretized kinetic equations, *Proc. Roy. Soc. Edinburgh*, **129A**, 1 (1999), 19-36.

[19] L. Bernis and L. Desvillettes, Propagation of singularities for classical solutions of the Vlasov-Poisson-Boltzmann equation, *Discrete Contin. Dyn. Syst.* 24 (2009), no. 1, 13-33.

[20] L. Boudin and L. Desvillettes, On the Singularities of the Global Small Solutions of the full Boltzmann Equation, *Monatschefte für Mathematik*, **131**, 2 (2000), 91-108.

[21] Carlo Cercignani, *The Boltzmann equation and its applications, Applied Mathematical Sciences*, vol. 67, Springer-Verlag, New York, 1988.

[22] Carlo Cercignani, Reinhard Illner, and Mario Pulvirenti, *The mathematical theory of dilute gases*, Applied Mathematical Sciences, vol. 106, Springer-Verlag, New York, 1994.

[23] S. Chapman and T. G. Cowling, *The mathematical theory of non-uniform gases*, Cambridge Univ. Press., London, 1970.

[24] Y. Chen, L. Desvillettes and L. He. Smoothing effects for classical solutions of the full Landau equation, *Arch. Ration. Mech. Anal.* 193 (2009), no. 1, 21-55.

[25] L. Desvillettes, On asymptotics of the Boltzmann equation when the collisions become grazing, *Transp. Th. Stat. Phys.*, **21**, 3 (1992), 259-276.

[26] L. Desvillettes, About the regularizing properties of the non-cut-off Kac equation, *Comm. Math. Phys.* 168 (1995), no. 2, 417-440.

[27] L. Desvillettes, About the Use of the Fourier Transform for the Boltzmann Equation, *Riv. Mat. Univ. Parma*, **7**, 2 (2003), 1-99 (special issue).

[28] L. Desvillettes and B. Wennberg, Smoothness of the solution of the spatially homogeneous Boltzmann equation without cutoff, *Comm. Partial Differential Equations* 29 (2004), no. 1-2, 133-155.

[29] L. Desvillettes and F. Golse, On a model Boltzmann equation without angular cutoff, *Diff. Int. equation*, **13**, 4-6 (2000), 567-594.

[30] Laurent Desvillettes and Clement Mouhot, Stability and uniqueness for the spatially homogeneous Boltzmann equation with long-range interactions, *Arch. Ration. Mech. Anal.* 193 (2009), no. 2, 227-253.

[31] R. DiPerna and P. L. Lions, On the Cauchy problem for the Boltzmann equation: Global existence and weak stability, *Ann. Math.*, **130**, 2 (1989), 312-366.

[32] Renjun Duan, Meng-Rong Li, and Tong Yang, Propagation of singularities in the solutions to the Boltzmann equation near equilibrium, *Math. Models Methods Appl. Sci.* 18 (2008), no. 7, 1093-1114.

[33] F. Golse, P. L. Lions, B. Perthame and R. Sentis, Regularity of the moments of the solution of a transport equation, *J. Funct. Anal.*, **76**, 1 (1988), 110-125.

[34] F. Golse and L. Saint-Raymond, The Navier Stokes limit of the Boltzmann equation for bounded collision kernels. *Invent. Math.* 155, 81-161 (2004)

[35] T. Goudon, On the Boltzmann equations and Fokker-Planck asymptotic: influence of grazing collisions. *J.Sta.Phys.* 89,3-4(1997), 751-776.

[36] Gressman and Robert M. Strain, Global Strong Solutions of the Boltzmann Equation without Angular Cut-off, preprint 2009.

[37] Gressman and Robert M. Strain, Global Classical Solutions of the Boltzmann Equation with Long-Range Interactions and Soft-Potentials, preprint 2010.

[38] Gressman and Robert M. Strain, Sharp anisotropic estimates for the Boltzmann collision operator and its entropy production, preprint 2010.

[39] Y. Guo, Classical solutions to the Boltzmann equation for molecules with an angular cutoff, *Arch. Ration. Mech. Anal.* 169 (2003), no. 4, 305-353.

[40] Y. Guo, The Boltzmann equation in the whole space, *Indiana Univ. Math. J.* 53 (2004), no. 4, 1081-1094.

[41] Z. Huo, Y. Morimoto, S. Ukai and T. Yang, Regularity of solutions for spatially homogeneous Boltzmann equation without angular cutoff. *Kinet. Relat. Models* 1 (2008), no. 3, 453-489.

[42] E. M. Lifshitz and L. P. Pitaevskii, *Physical kinetics*, Perg. Press., Oxford, 1981.

[43] P. Lions, Regularite et compacite pour des noyaux de collision de Boltzmann sans troncature angulaire, *C. R. Acad. Sci. Paris Serie. Math.* 326 (1998), no. 1, 37-41.

[44] P.-L. Lions and N. Masmoudi, From the Boltzmann equations to the equations of incompressible fluid mechanics, I. *Archive Rat. Mech. Anal.* 158 (2001), 173-193.

[45] P.-L. Lions and N. Masmoudi, From the Boltzmann equations to the equations of incompressible fluid mechanics II. *Archive Rat. Mech. Anal.* 158 (2001), 195-211.

[46] T. Liu, T. Yang, and Shih-Hsien Yu, Energy method for Boltzmann equation, *Phys. D* 188 (2004), no. 3-4, 178-192.

[47] X. Lu, A direct method for the regularity of the gain term in the Boltzmann equation, *J. Math. Anal. Appl.*, **228**, 2 (1998), 409-435.

[48] Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang, Regularity of solutions to the spatially homogeneous Boltzmann equation without angular cutoff, *Discrete Contin. Dyn. Syst.* 24 (2009), no. 1, 187-212.

[49] C. Villani, On a new class of weak solutions for the spatially homogeneous Boltzmann and Landau equations. *Arch. Rat. Mech. Anal.* 143 (1998), 273-307.

[50] C. Villani, Regularity estimates via the entropy dissipation for the spatially homogeneous Boltzmann equaiton without cut-off, *Rev.Mat. Iberoam.* 15, 2 (1999), 335-352.

[51] C. Villani, A review of mathematical topics in collisional kinetic theory, North-Holland, Amsterdam, Handbook of mathematical fluid dynamics, Vol. I, 2002, pp. 71-305.

[52] B. Wennberg, Regularity in the Boltzmann equation and the Radon transform, *Comm. Partial Differential Equations*, **19**, 11-12 (1994), 2057-2074.

(Y. Chen) Department of Mathematics, Beijing Institute of Technology, 100081 Beijing, P. R. China.
  E-mail address: yeminchen@bit.edu.cn

(L. He) Department of Mathematical Sciences, Tsinghua University, Beijing 100084, P. R. China.
  E-mail address: lbhe@math.tsinghua.edu.cn